Optimization of Switched Linear Systems over Non-Stationary Wireless Channels

Mark Eisen Konstantinos Gatsis George J. Pappas Alejandro Ribeiro

***Abstract*—This paper considers the switched linear system for a series of control systems that are closed over a wireless channel that is unknown and non-stationary. The goal is to find power allocation policies that minimize a long term quadratic cost of all state variables by closing the loop sufficiently often while satisfying a budget constraint. The problem is formulated through duality as a stochastic optimization problem. Because the channel distribution is not know, and empirical risk is solved to approximate the problem. As the channel varies over time, the empirical risk minimization must be continuously approximated. The second order Newton's method is presented as an effective approach to find approximated allocation policies over as the channel varies because of its quadratic convergence and the closeness of consecutive solutions. Under certain conditions on the sampling size and rate of channel variation, we establish a control performance suboptimality for each time epoch and subsequently demonstrate long term stability of the states. We additionally provide a numerical experiment that illustrates the theoretical results.**

***Index Terms*—wireless control system, resource allocation, second order method, non-stationary channel**

I. Introduction

The recent advances in Internet-of-Things has provided further motivation for intelligent design of wireless control systems. In particular, it is of interest in studying or designing the communication parameters in relation to the control performance or stability of the system. Such analyses include relating the stability of the plant to the packet drop rate of the channel, [1], [2] and channel capacity [3], [4]. Another area of interest is designing communication resource allocation policies to optimize control performance in, e.g., power allocation over fading channels [5], [6], or in event-triggered control [7], [8]. All of these approaches, however, require the wireless channel itself to be adequately modeled.

This second area of optimizing control performance through design of communication parameters can still nonetheless be studied without a model through successive sampling of the channel. Existing works use sampling-based optimization methods to optimize performance in both wireless control systems [9], [10] and general wireless systems [11]. These approaches however must assume that the wireless channel is stationary. In this work, we design a sampling-based optimization that can quickly find optimal operating points in non-stationary, or time varying, channels. This is done by exploiting both the quadratic convergence of second order optimization methods as well as the approximation error incurred through replacing a model with channel samples. The authors previously studied the dual suboptimality of such an approach in [12], [13], and here study its performance on the switched linear system common in wireless systems by characterizing the control performance and establishing stability. We focus in particular of switched linear system model for communications, also studied in, e.g., [9], [14], [15].

The goal of this paper is to design resource allocation policies that can stabilize the switched linear system over a random and time-varying wireless channel, while constrained by a resource budget. The channel is unknown and can only be observed through samples across time. We demonstrate in Section II that the optimal power allocations of the control system can be modeled with a stochastic optimization problem. When channel samples are used in place of the model, this becomes an empirical risk minimization (ERM) problem (Section III). To find optimal allocation policies at each point in time as the channel varies, we present a second order optimization method that can learn statistically accurate policies with a single iteration (Section III-A). We establish conditions under which the system exhibits small suboptimality of control performance and stays in the stable region over all time (Section IV). A numerical simulation is presented in Section V. Proofs not present in this paper are found in [16].

II. Problem Formulation

We consider a setting of m independent control systems labeled $i = 1, \dots, m$, each of which closes its control loop by sending state information to its controller over a wireless channel. Depending upon whether or not the message is successfully decoded and received at the controller, the system operates in either closed or open loop. Specifically, each node i has a state variable $x_t^i \in \mathbb{R}$ that dynamically evolves over time t with the following *switched dynamical system*

$$x_{t+1}^i = \begin{cases} A_c^i x_t^i + w_t^i & \text{if loop closes} \\ A_o^i x_t^i + w_t^i & \text{otherwise} \end{cases} \tag{1}$$

In (1), $A_c^i < 1$ and $A_o^i > 1$ are the closed and open loop dynamics of system i and w_t^i is some zero-mean i.i.d. disturbance process with variance W^i. These systems arise in cases where the control inputs u_t^i are expressed as Kx_t^i for some K (e.g. LQR control). Note that the system in (1) and proceeding derivations can be extended to multidimensional states as well, but is kept in scalar form here for simplicity.

The closing of the loop is determined by the signal-to-noise ratio (SNR) at the receiver, which is the product of the transmit power $p^i \in \mathbb{R}_+$ and channel condition $h^i \in \mathbb{R}_+$ of system i. The channel fading conditions change randomly over time in an unpredictable manner [17, Ch. 3], and are traditionally modeled as i.i.d. random variables taken from distribution

$\mathcal{H}$. The conditions of the wireless channel distribution $\mathcal{H}$ may slowly vary in a non-stationary manner, and is therefore indexed as $\mathcal{H}_k$ with channel states h_k^i for time epoch k, which will in general encompass many time steps t.

We are interested in a quadratic system cost objective that measures the cumulative growth of x_t^i over all time, i.e.

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}(x_t^i)^2 \tag{2}$$

Considering the switched system dynamics in (1) during epoch k, the system state decreases when the system is in closed loop and increases in open loop. We consider a function $q(h_k^i, p^i)$ that, given a current channel state and transmit power, gives the probability of successful transmission and decoding of the transmitted packet [5], [6]. Given such a function, the goal is to minimize the aggregate cost in (2) across all systems $i = 1, \ldots, m$ while maintaining a total power budget. As such, we can determine a transmit power *policy* for each system, $p^i(h_k^i) : \mathbb{R} \to \mathbb{R}_+$ that selects a transmission power given a current channel state h_k^i. We define the ergodic average over all channel states as

$$y_k^i := \mathbb{E}_{h_k^i} \left\{ q(h_k^i, p^i(h_k^i)) \right\}. \tag{3}$$

From the ergodic average definition in (3), we can reformulate (2) without the limit in time. Specifically, it can be seen that by using the system parameters in (1), the quadratic cost can be equivalently viewed as a function of y_k^i along with the system parameters. We can consequently formulate an optimization problem that minimizes (2) for all systems while constraining the total expected power $\sum_i \mathbb{E}_h p^i(h_k^i)$ to be less than a budget P_0. Defining the boldface notation for any vector $\mathbf{z} \in \mathbb{R}^m := [z^1; z^2; \ldots; z^m]$, we represent optimal power allocation policies $\mathbf{p}_k^*(\mathbf{h})$ and associated ergodic averages $\mathbf{y}_k^*$ to be the solution to the following optimization problem.

$$[\mathbf{p}_k^*(\mathbf{h}), \mathbf{y}_k^*] := \operatorname*{argmin}_{\mathbf{p}, \mathbf{y} \in \mathbb{R}^m} \sum_{i=1}^{m} \frac{W^i}{1 - y^i (A_c^i)^2 - (1 - y^i)(A_o^i)^2} \tag{4}$$
$$\text{s.t. } \mathbf{y} \leq \mathbb{E}_{\mathbf{h}_k} \mathbf{q}(\mathbf{h}_k, \mathbf{p}(\mathbf{h}_k)), \quad \mathbf{1}^T \mathbb{E}_{\mathbf{h}_k} \mathbf{p}(\mathbf{h}_k) \leq P_0.$$

The objective in (4) is indeed an equivalent formulation of the cost in (2)—see, e.g. [16, Example 1]—while we also point out that the equivalence relation in (3) can be relaxed to an inequality constraint in (4). Note that this problem is, in its current form, difficult to solve due to both the possible nonconvexity of the first constraint and the fact that the policy $\mathbf{p}_k^*(\mathbf{h})$ is an infinite dimensional variable. However, an important result in [18] shows the problems of this form exhibit zero duality gap and can thus be solved in the dual domain. Using standard techniques of Lagrangian duality (see, e.g. [19]), the dual problem can be written as

$$\{\boldsymbol{\mu}_k^*, \lambda_k^*\} := \operatorname*{argmax}_{\boldsymbol{\mu}, \lambda \geq 0} L_k(\boldsymbol{\mu}, \lambda) := \operatorname*{argmax}_{\boldsymbol{\mu}, \lambda \geq \mathbf{0}} \mathbb{E}_{\mathbf{h}_k} f(\boldsymbol{\mu}, \lambda, \mathbf{h}_k), \tag{5}$$

where $f(\boldsymbol{\mu}, \lambda, \mathbf{h}_k) := \min_{\mathbf{p}, \mathbf{y}} \left\{ \sum_{i=1}^{m} \frac{W^i}{1 - y^i (A_c^i)^2 - (1 - y^i)(A_o^i)^2} \right.$

$$\left. + \boldsymbol{\mu}^T (\mathbf{y} - \mathbf{q}(\mathbf{h}_k, \mathbf{p}(\mathbf{h}_k))) + \lambda (\mathbf{1}^T \mathbf{p}(\mathbf{h}_k) - P_0) \right\}.$$

The dual problem in (5) in inherently easier to solve than (4) because it is convex and optimizes over low-dimensional dual variables $\boldsymbol{\mu} \in \mathbb{R}_+^m$ and $\lambda \in \mathbb{R}_+$. Because of the property of zero duality gap, $\mathbf{p}^*(\mathbf{h}_k)$ and be recovered from $\{\boldsymbol{\mu}^*, \lambda\}$ as the arguments of the minimization operator in $f(\boldsymbol{\mu}^*, \lambda^*, \mathbf{h}_k)$.

III. Empirical risk minimization

The dual loss function $L_k(\boldsymbol{\mu}, \lambda)$ is a statistical loss function over the channel distribution $\mathcal{H}_k$. As the channel distributions are in general difficult to model and will be varying an a non-stationary manner over time epochs, we can substitute the statistical loss function in (5) with an *empirical risk* function, which replaces the expectation operator with an empirical average over N channel samples $\mathbf{h}_k^1, \ldots, \mathbf{h}_k^N$. This is a common substitution in machine learning and is referred to as empirical risk minimization (ERM). The empirical loss function in epoch k is formally defined as

$$\hat{L}_k(\boldsymbol{\mu}, \lambda) := \frac{1}{N} \sum_{l=1}^{N} f(\boldsymbol{\mu}, \lambda, \mathbf{h}_k^l). \tag{6}$$

Substituting the empirical risk $\hat{L}_k$ into the dual problem in (5) results in a convex and deterministic program, solvable with a wide array of convex finite-sum optimization methods. Indeed, this substitution only serves as an approximation of the true dual problem, although one that can be made stronger with the number of samples taken. We define a quantity called the *statistical accuracy* of $\hat{L}_k(\boldsymbol{\mu}, \lambda)$, which is the max difference between $\hat{L}_k(\boldsymbol{\mu}, \lambda)$ and $L_k(\boldsymbol{\mu}, \lambda)$ across the domain. This bound is well-studied for simple i.i.d. samples in machine learning literature and is of the order of $\mathcal{O}(1/\sqrt{N})$ for N samples [20]. In this paper we define V_N to be the statistical accuracy.

Remark 1: We note that drawing N new samples at each epoch form $\hat{L}_k$ may be limiting in practical scenarios. For such cases, an alternative sampling approach is to keep $(M-1)N/M$ samples previously drawn the window $\mathcal{H}_{k-M+1}, \ldots, \mathcal{H}_{k-1}$ and draw only N/M new samples from $\mathcal{H}_k$. The exact bounds on the statistical accuracy achieved by $\hat{L}_k$ in this non-i.i.d. case are not well studied, so is not considered in depth in this work, but can reduce the sampling complexity necessary at each epoch when consecutive distributions $\mathcal{H}_{k-M+1}, \ldots, \mathcal{H}_k$ are close.

Given that the maximization of the ERM function will only maximize the dual function to within the statistical accuracy, we may consider any additional bias of $\mathcal{O}(V_N)$ is permissible. Therefore we augment the ERM function in (6) with two additional regularization terms that induce desirable properties to the maximization problem. Firstly, we add the regularization term $\alpha V_N/2 \|\boldsymbol{\mu}\|^2$ to the empirical risk in (6) to make the problem strongly convex. Secondly, we remove the non-negativity constraint of the dual parameters in (5) by adding a logarithmic barrier regularizer. To preserve smoothness for small $\boldsymbol{\mu}$, however, we use an ϵ-threshold log function, i.e.

$$\log_\epsilon(\mathbf{z}) := \begin{cases} \log(\mathbf{z}) & \mathbf{z} \geq \epsilon \\ \ell_{2,\epsilon}(\mathbf{z} - \epsilon) & \mathbf{z} < \epsilon, \end{cases} \tag{7}$$

where $\ell_{2,\epsilon}(\mathbf{z})$ is a second order Taylor series expansion of $\log(\mathbf{z})$ centered at ϵ for some small $0 < \epsilon < 1$. The second regularization term $-\beta V_N \mathbf{1}^T \log_\epsilon \boldsymbol{\mu}$ is also added, resulting in the regularized empirical risk function

$$\hat{R}_k(\boldsymbol{\mu}, \lambda) := \frac{1}{N} \sum_{l=1}^{N} f(\boldsymbol{\mu}, \lambda, \mathbf{h}^l) + \frac{\alpha V_N}{2} \|\boldsymbol{\mu}\|^2 - \beta V_N \mathbf{1}^T \log_\epsilon \boldsymbol{\mu} + \frac{\alpha V_N}{2} \lambda^2 - \beta V_N \log_\epsilon \lambda. \tag{8}$$

At each epoch k, we are interested in finding the approximate optimal dual parameters, defined as

$$\{\hat{\boldsymbol{\mu}}_k^*, \hat{\lambda}_k^*\} := \underset{\boldsymbol{\mu}, \lambda}{\operatorname{argmax}} \, \hat{R}_k(\boldsymbol{\mu}, \lambda). \tag{9}$$

Both the quadratic and log-barrier regularizers, when scaled by V_N, are known to introduce biases of this order—see, e.g., [19], [21] for details—and we can say the solutions in (9) are within the statistical accuracy of the solutions in (5).

Continuously solving (9) at every epoch can be costly and time-consuming, and therefore infeasible in the wireless dynamical systems we are interested in. However, in the following section we demonstrate how the quadratic convergence of Newton's method make it possible to instantaneously find approximate solutions to (9) at each epoch k.

A. Solving in Non-Stationary Channels

To solve (9) at each epoch k, a naive approach would be to draw N samples from the new channel distribution $\mathcal{H}_k$ and then solve the optimization problem directly using a standard convex optimization method—at potentially significant computation time and cost. However, there are two observations that inspire a more direct approach to finding the optimal dual parameters. The first is that the solution to (9) only solves the true problem (5) up to statistical accuracy V_N, making it unnecessary to solve (9) to within more than V_N. We therefore only look for V_N-optimal solutions to (9). The second observation is that the risk functions for consecutive epochs $\hat{R}_k$ and $\hat{R}_{k+1}$ differ only in the channel distributions $\mathcal{H}_k$ and $\mathcal{H}_{k+1}$ from which the samples are drawn. Assuming the channel distributions evolve in a smooth manner, $\mathcal{H}_k$ and $\mathcal{H}_{k+1}$ will be close to each other, as will $\{\hat{\boldsymbol{\mu}}_k^*, \hat{\lambda}_k^*\}$ and $\{\hat{\boldsymbol{\mu}}_{k+1}^*, \hat{\lambda}_{k+1}^*\}$. Therefore, starting from V_N-optimal dual parameters at epoch k, we should not be far from V_N- optimal dual parameters at epoch $k+1$.

Newton's method is well suited for finding approximate solutions in the non-stationary setting across time epochs due its property of local quadratic convergence—see, e.g. [19]. This means that, when close to the next optimal solution, Newton's method can reach near-optimal iterates in just a single update. To define the update, consider the gradient $\nabla \hat{R}_k(\boldsymbol{\mu}, \lambda)$ and Hessian $\nabla^2 \hat{R}_{k+1}(\boldsymbol{\mu}, \lambda)$ of the regularized empirical risk function. At epoch $k+1$, we find a new dual parameter estimate using the previous epoch's estimate with the standard Newton update formula

$$\begin{bmatrix} \boldsymbol{\mu}_{k+1} \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}_{k} \\ \lambda_{k} \end{bmatrix} - \begin{bmatrix} \nabla^2_{\boldsymbol{\mu}\boldsymbol{\mu}} \hat{R}_{k+1}(\boldsymbol{\mu}_k) & \nabla^2_{\boldsymbol{\mu}\lambda} \hat{R}_{k+1}(\boldsymbol{\mu}_k) \\ \nabla^2_{\boldsymbol{\mu}\lambda} \hat{R}_{k+1}(\boldsymbol{\mu}_k) & \nabla^2_{\lambda\lambda} \hat{R}_{k+1}(\boldsymbol{\mu}_k) \end{bmatrix}^{-1} \begin{bmatrix} \nabla_{\boldsymbol{\mu}} \hat{R}_{k+1}(\boldsymbol{\mu}_k) \\ \nabla_{\lambda} \hat{R}_{k+1}(\boldsymbol{\mu}_k) \end{bmatrix} \tag{10}$$

At each epoch, using the dual parameter $\boldsymbol{\mu}_k$, we can then recover a near optimal power allocation policy $\mathbf{p}_k(\mathbf{h}_k)$ as

$$\mathbf{p}_k(\mathbf{h}_k) = \underset{\mathbf{p}}{\operatorname{argmin}} \left\{ -\boldsymbol{\mu}_k^T \mathbf{q}(\mathbf{h}_k, \mathbf{p}(\mathbf{h}_k)) + \lambda_k \mathbf{1}^T \mathbf{p}(\mathbf{h}_k) \right\}. \tag{11}$$

The complete algorithm across all time is then presented in Algorithm 1. After preliminaries and initialization in Steps 1-4, the backtracking loop starts in Step 5. Each iteration begins in Step 6 with the the drawing of N samples from the new channel distribution $\mathcal{H}_{k+1}$ to form $\hat{R}_{k+1}$. The gradient $\nabla \hat{R}_{k+1}$ and Hessian $\mathbf{H}_{k+1}$ of the regularized dual loss function are computed in Step 7, after which the Newton step is taken to update $\boldsymbol{\mu}_{k+1}$ in Step 8. In Step 9, the near-optimal resource allocation policy $\mathbf{p}_{k+1}(\mathbf{h}_{k+1})$ is determined using the updated dual variable. We include a backtracking step for the sample draw N in Step 10 to ensure the new iterate $\boldsymbol{\mu}_{k+1}$ is within the statistical accuracy V_N of $\hat{R}_{k+1}$, as verified in Step 11, to adapt to unknown system parameters.

Algorithm 1

1: **Parameters:** Sample size increase constants $N_0 \geq 1$ backtracking params $0 < \delta < 1$, α, β.
2: **Input:** Initial sample size $N = N_0$ and argument $\boldsymbol{\mu}_0$ $\|\nabla \hat{R}_0(\boldsymbol{\mu}_0)\| < (\sqrt{2\alpha}) V_N$
3: **for** $k = 0, 1, 2, \ldots$ **do** {main loop}
4: Reset factor $N = N_0$.
5: **repeat** {sample size backtracking loop}
6: Draw N samples from $\mathcal{H}_{k+1}$.
7: Compute gradient $\nabla \hat{R}_{k+1}(\boldsymbol{\mu}_k)$, Hessian $\nabla^2 \hat{R}_{k+1}(\boldsymbol{\mu}_k)$.
8: Update [cf. (10)]: $\boldsymbol{\mu}_{k+1} = \boldsymbol{\mu}_k - \nabla^2 \hat{R}_{k+1}^{-1}(\boldsymbol{\mu}_k) \nabla \hat{R}_{k+1}(\boldsymbol{\mu}_k)$
9: Determine power allocation [cf. (11)]: $\mathbf{p}_{k+1}(\mathbf{h}_{k+1}) = \operatorname{argmin}_{\mathbf{p}} \left\{ -\boldsymbol{\mu}_{k+1}^T \mathbf{q}(\mathbf{h}_{k+1}, \mathbf{p}(\mathbf{h}_{k+1})) + \lambda_{k+1} \mathbf{1}^T \mathbf{p}(\mathbf{h}_{k+1}) \right\}$
10: Backtrack sample draw $N = \delta N$.
11: **until** $\|\nabla \hat{R}_n(\boldsymbol{\mu}_{k+1})\| < (\sqrt{2\alpha}) V_N$
12: **end for**

IV. Theoretical Analysis

In this section, we provide a theoretical analysis of the use of Newton's method on the switched linear system in a non-stationary distribution. We specifically make characterizations of the suboptimality and constraint violation of the dual parameters $\boldsymbol{\mu}_k, \lambda_k$ found with the Newton update in (10) to demonstrate stability of the resulting system. We first state a series of assumptions made in our analysis.

Assumption 1: The statistical loss functions $f(\boldsymbol{\mu}, \lambda, \mathbf{h}_k)$ are self-concordant with respect to $\boldsymbol{\mu}$ and λ and have gradients $\nabla f(\boldsymbol{\mu}, \lambda, \mathbf{h}_k)$ that are Lipschitz continuous with constant Δ.

Assumption 2: The difference between the gradients of the empirical loss $\hat{L}_k$ and the statistical average loss L_k is bounded by $V_N^{1/2}$ for all $\{\boldsymbol{\mu}, \lambda\}$ and k with high probability,

$$\sup_{\boldsymbol{\mu},\lambda} \|\nabla L_k(\boldsymbol{\mu}, \lambda) - \nabla \hat{L}_k(\boldsymbol{\mu}, \lambda)\| \leq V_N^{1/2}, \qquad \text{w.h.p.} \quad (12)$$

Assumption 3: The difference between two successive expected loss $L_k(\boldsymbol{\mu}, \lambda) = \mathbb{E}_{h_k} f(\boldsymbol{\mu}, \mathbf{h}_k)$ and $L_{k+1}(\boldsymbol{\mu}, \lambda) = \mathbb{E}_{h_{k+1}} f(\boldsymbol{\mu}, \mathbf{h}_{k+1})$ and the difference between gradients are bounded respectively by a bounded sequence of constants $\{D_k\}, \{\bar{D}_k\} \geq 0$ for all $\{\boldsymbol{\mu}, \lambda\}$,

$$\sup_{\boldsymbol{\mu},\lambda} |L_k(\boldsymbol{\mu}, \lambda) - L_{k+1}(\boldsymbol{\mu}, \lambda)| \leq D_k, \quad (13)$$

$$\sup_{\boldsymbol{\mu},\lambda} \|\nabla L_k(\boldsymbol{\mu}, \lambda) - \nabla L_{k+1}(\boldsymbol{\mu}, \lambda)\| \leq \bar{D}_k. \quad (14)$$

Assumption 4: For all epochs k, the problem in (4) under distribution $\mathcal{H}_k$ is strictly feasible. Also, the optimal dual variable is bounded as $\|[\boldsymbol{\mu}_k^*, \lambda_k^*]\| \leq \hat{\mathcal{K}}$.

Assumption 1, in addition to self concordance (i.e. $|f'''(\boldsymbol{\mu}, \lambda)|_i \leq 2 f''(\boldsymbol{\mu}, \lambda)_i^{3/2}$ for all dimensions i), implies that the regularized empirical loss gradients $\nabla \hat{R}_k$ are Lipschitz continuous with constant $\Delta + cV_N$ where $c := \alpha + \beta/\epsilon^2$ and the function $\hat{R}_k$ is strongly convex with constant αV_N. Assumption 2 can be established through the law of large numbers while Assumption 3 effectively provides a limit on the rate at which the channel evolves between epochs.

Using these properties along with the properties of Newton's method, we characterize the control performance suboptimality of the dual iterates $\{\boldsymbol{\mu}_k, \lambda_k\}$ generated by the Newton update in (10). We first use the following Lemma to establish conditions under which (10) achieves statistical accuracy at each epoch with respect to the regularized empirical risk.

Lemma 1: Consider Newton's method defined in (10). Further consider the variable $\{\boldsymbol{\mu}_k, \lambda_k\}$ as a V_N-optimal solution of the loss $\hat{R}_k$, and suppose Assumptions 1-3 hold. If

$$\left(\frac{2(\Delta + cV_N)V_N}{\alpha V_N}\right)^{1/2} + \frac{2V_N^{1/2} + \bar{D}_k}{(\alpha V_N)^{1/2}} < \frac{1}{4} \quad (15)$$

$$144(5V_N + 2D_k)^2 \leq V_N \quad (16)$$

are satisfied, then the variable $\boldsymbol{\mu}_{k+1}$ computed from (10) has the suboptimality of V_N with high probability, i.e.,

$$\hat{R}_{k+1}(\boldsymbol{\mu}_{k+1}) - \hat{R}_{k+1}^* \leq V_N, \qquad \text{w.h.p.} \quad (17)$$

The expressions in (15) and (16) provide conditions on V_N (controlled by sampling rate N), D_k and $\bar{D}_k$, such that Newton's method produces V_N-accurate dual variables for each k. Because these parameters may not be known in practice, we include a backtracking step (as done in Algorithm 1) to control parameters N to achieve statistical accuracy. We now derive an important bound on the suboptimality of the control performance metric used in the objective of (4) (itself a reformulation of (2)).

Theorem 1: Consider $\boldsymbol{\mu}_k$ to be a V_N-optimal minimizer of R_k. Define $J(\mathbf{y}_k)$ to be the control performance objective in (4). There exists a finite constants C and c such that the control performance sub-optimality can be upper bounded as

$$J(\mathbf{y}_k) - J(\mathbf{y}_k^*) \leq CV_N + c. \quad (18)$$

Theorem 1 establishes a control performance sub-optimality on the order of V_N and a constant of the ergodic averages $\mathbf{y}_k$ generated by the Newton update. We can relate this sub-optimality back to the switched linear system in (1) and use it to establish a stability result. Recall that the open loop gain $A_o > 1$ can cause the system to grow unstably if the system is not closed sufficiently often. With the following corollary, we establish that the ergodic averages $\mathbf{y}_k$ keep the system stable over all time and all epochs.

Corollary 1: Consider $\mathbf{y}_k$ to be the ergodic average variables generated by the Newton update. These averages keep the state x_t^i governed by (1) finite for all t over non-stationary channel.

Proof: In (18) we have control performance subobtimality is bounded by a term proportional to V_N. If we assume $J(\mathbf{y}_k^*)$ is finite for all epochs k, it follows that $J(\mathbf{y}_k)$ is also finite. Referring back to the objective in (4) that $J(\mathbf{y}_k)$ represents, this implies that $y_k^i (A_c^i)^2 + (1 - y_k^i)(A_o^i)^2 \leq \rho < 1$ at all epochs k and systems i for some ρ. As y_k^i is an ergodic average, the variance of the system state satisfies the recursive formula

$$\mathbb{E}(x_{t+1}^i)^2 = y_k^i\,(A_c^i)^2\,\mathbb{E}x_t^2 + (1 - y_k^i)\,(A_o^i)^2\,\mathbb{E}(x_t^i)^2 + W^i \quad (19)$$

Substituting $y_k^i (A_c^i)^2 + (1 - y_k^i)(A_o^i)^2 \leq \rho$ into (19), we obtain

$$\mathbb{E}(x_{t+1}^i)^2 \leq \rho \mathbb{E}(x_t^i)^2 + W^i. \quad (20)$$

Operating recursively and using the geometric series substitution, we can rewrite (20) as

$$\begin{aligned} \mathbb{E}(x_{t+1}^i)^2 &\leq \rho^{t+1}\mathbb{E}(x_0^i)^2 + \sum_{s=0}^{t} \rho^t W^i \\ &= \rho^{t+1}\mathbb{E}(x_0^i)^2 + W^i \frac{1 - \rho^{t+1}}{1 - \rho}. \end{aligned} \quad (21)$$

As both terms on the right hand side of (21) are finite, we can conclude that the state variables remain bounded for all t in the non-stationary channel. ■

V. Simulation Results

For our simulations, the open and closed loop control gains A_o^i and A_c^i are chosen between $[1.1, 1.5]$ and $[0, 0.8]$, respectively, for $m = 4$ systems. The probability of successful transmission is modeled as a negative exponential function of both the power and channel state, $q(h^i, p^i(h^i) := 1 - e^{-h^i p^i(h^i)}$, while the channel states at epoch k are drawn from an exponential distribution with mean u_k. The time-varying channel varies u_k over epochs and draw $N = 200$ samples.

To demonstrate the ability of Newton's method to find approximately optimal power allocation as the channel distribution varies over time, we perform Algorithm 1 using the above parameters. In Figure 1 we show the path of the resulting control performance at each epoch k using the dual

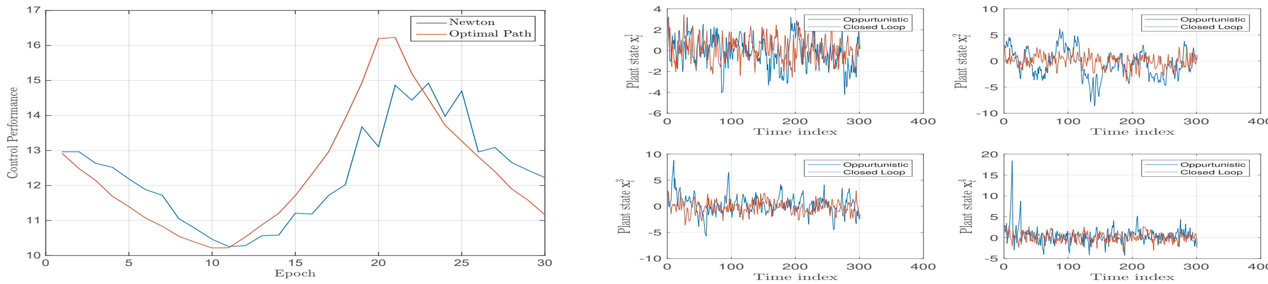

Fig. 1: (left) Convergence paths of optimal control performance vs. that generated by proposed method for time-varying $\mathcal{H}_k$. Newton's method is able to find an approximately optimal value for the dual variable at each iteration. (right) State evolution for 4 systems using opportunistic policy found with proposed method over time-varying channel.

parameters found with (10). The red line of each figure plots the optimal values for the current distribution parameter u_k as it changes with k. The blue line, alternatively, plots the values generated by Newton's method over epochs. The channel evolves at each iteration by a fixed rate $u_{k+1} = u_k \pm r$ for some rate r. Observe that within some small error that Newton's method is indeed able to quickly and approximately find each new solution as the channel varies over time.

With the found power allocation policies, we simulate the resulting dynamical system. Figure 1 shows the resulting state evolution of x_t^i for each of the 4 states. The blue curve shows the process using the opportunistic transmission policy from Newton's method, while the red curve shows the process when the loop is always closed. Here, we observe that while there are some instances when the state variable grows large when the system is in open loop, overall the system remains stable.

VI. Conclusion

In this paper we develop a method of determining near-optimal power allocation policies in a switched linear control system over a non-stationary wireless channel. We apply Lagrangian duality to reformulate the problem as a statistical loss problem, which can be further approximated using samples as empirical risk minimization. The quadratic convergence of Newton's method allows one to solve consecutive ERM problems over time epochs with single updates. We establish formal conditions for this and characterize the suboptimality and stability in the switched dynamical system.

References

[1] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S.S. Sastry, "Foundations of control and estimation over lossy networks," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 163–187, 2007.

[2] Joao P Hespanha, Payam Naghshtabrizi, and Yonggang Xu, "A survey of recent results in networked control systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138–162, 2007.

[3] S. Tatikonda and S. Mitter, "Control under communication constraints," *IEEE Transactions on Automatic Control*, vol. 49, no. 7, pp. 1056–1068, 2004.

[4] A. Sahai and S. Mitter, "The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link&# 8212; part i: Scalar systems," *IEEE Transactions on Information Theory*, vol. 52, no. 8, pp. 3369–3395, 2006.

[5] Konstantinos Gatsis, Alejandro Ribeiro, and George J. Pappas, "Optimal power management in wireless control systems," *IEEE Transactions on Automatic Control*, vol. 59, no. 6, pp. 1495–1510, 2014.

[6] Daniel E Quevedo, Anders Ahlén, Alex S Leong, and Subhrakanti Dey, "On Kalman filtering over fading wireless channels with controlled transmission powers," *Automatica*, vol. 48, no. 7, pp. 1306–1316, 2012.

[7] José Araújo, Manuel Mazo, Adolfo Anta, Paulo Tabuada, and Karl H Johansson, "System architectures, protocols and algorithms for aperiodic wireless control systems," *IEEE Transactions on Industrial Informatics*, vol. 10, no. 1, pp. 175–184, 2014.

[8] Mohammad H Mamduhi, Domagoj Tolić, Adam Molin, and Sandra Hirche, "Event-triggered scheduling for stochastic multi-loop networked control systems with packet dropouts," in *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*. IEEE, 2014, pp. 2776–2782.

[9] Konstantinos Gatsis, Miroslav Pajic, Alejandro Ribeiro, and George J. Pappas, "Opportunistic control over shared wireless channels," *IEEE Transactions on Automatic Control*, vol. 60, no. 12, pp. 3140–3155, December 2015.

[10] Konstantinos Gatsis, Alejandro Ribeiro, and George J. Pappas, "Random access design for wireless control systems," *Automatica*, 2018, To appear. Available on Arxiv.

[11] Tianyi Chen, Aryan Mokhtari, Xin Wang, Alejandro Ribeiro, and Georgios B Giannakis, "Stochastic averaging for constrained optimization with application to online resource allocation," *IEEE Transactions on Signal Processing*, vol. 65, no. 12, pp. 3078–3093, 2017.

[12] Mark Eisen, Konstantinos Gatsis, George J. Pappas, and Alejandro Ribeiro, "Learning in non-stationary wireless control systems via newton's method," in *American Control Conference (ACC)*, 2018, vol. (to appear), Available at http://www.fling.seas.upenn.edu/ maeisen/wiki/acc2s.pdf.

[13] Mark Eisen, Konstantinos Gatsis, George J. Pappas, and Alejandro Ribeiro, "Learning statistically accurate resource allocations in non-stationary wireless systems," in *International Conference on Acoustics Speech Signal Process. (ICASSP)*, 2018, vol. (to appear), Available at http://www.fling.seas.upenn.edu/ maeisen/wiki/icassp2s.pdf.

[14] Daniel Liberzon, "Finite data-rate feedback stabilization of switched and hybrid linear systems," *Automatica*, vol. 50, no. 2, pp. 409–420, 2014.

[15] Guosong Yang and Daniel Liberzon, "Feedback stabilization of switched linear systems with unknown disturbances under data-rate constraints," *IEEE Transactions on Automatic Control*, 2017.

[16] Mark Eisen, Konstantinos Gatsis, George J. Pappas, and Alejandro Ribeiro, "Learning in wireless control systems over non-stationary channels," 2018, Available at http://www.fling.seas.upenn.edu/ maeisen/wiki/wirelesscontrol2.pdf.

[17] Andrea Goldsmith, *Wireless communications*, Cambr. Univ. Press, 2005.

[18] Alejandro Ribeiro, "Ergodic stochastic optimization algorithms for wireless communication and networking," *IEEE Transactions on Signal Processing*, vol. 58, no. 12, pp. 6369–6386, 2010.

[19] Stephen Boyd and Lieven Vandenberghe, *Convex optimization*, Cambridge university press, 2004.

[20] Vladimir Vapnik, *The nature of statistical learning theory*, Springer science & business media, 2013.

[21] Shai Shalev-Shwartz, Ohad Shamir, Nathan Srebro, and Karthik Sridharan, "Learnability, stability and uniform convergence," *Journal of Machine Learning Research*, vol. 11, no. Oct, pp. 2635–2670, 2010.