Minimal Actuator Placement With Bounds on Control Effort

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***Abstract*—We address the problem of minimal actuator placement in a linear system subject to an average control energy bound. First, following the recent work of Olshevsky, we prove that this is NP-hard. Then, we provide an efficient algorithm which, for a given range of problem parameters, approximates up to a multiplicative factor of $O(\log n)$, with n being the network size, any optimal actuator set that meets the same energy criteria; this is the best approximation factor one can achieve in polynomial time in the worst case. Moreover, the algorithm uses a perturbed version of the involved control energy metric, which we prove to be supermodular. Next, we focus on the related problem of cardinality-constrained actuator placement for minimum control effort, where the optimal actuator set is selected so that an average input energy metric is minimized. While this is also an NP-hard problem, we use our proposed algorithm to efficiently approximate its solutions as well. Finally, we run our algorithms over large random networks to illustrate their efficiency.**

***Index Terms*—Controllability energy metrics, input placement, leader selection, minimal network controllability, multiagent networked systems.**

I. Introduction

DURING the past decade, an increased interest in the analysis of large-scale systems has led to a variety of studies that range from the mapping of the human's brain functional connectivity to the understanding of the collective behavior of animals, and the evolutionary mechanisms of complex ecological systems [1]–[4]. At the same time, control scientists have developed methods for the regulation of such complex systems, with notable examples in [5] for the control of biological systems, [6] for the regulation of brain and neural networks, [7] for robust information spread over social networks, and [8] for load management in the smart grid.

On the other hand, the large size of these systems as well as the need for low-cost control, has made the identification of a small fraction of their states to steer them around the entire space, an important problem [9]–[12]. This is a task of formidable complexity; indeed, it is shown in [9] that finding a small number of actuators, so that a linear system is controllable, is NP-hard. However, mere controllability is of little value if the required input energy for the desired transfers is exceedingly high when, for example, the controllability matrix is close to singularity [13]. Therefore, by choosing input states to ensure controllability alone, one may not achieve cost-effective control for the system.

In this paper, we address this important requirement by providing efficient approximation algorithms to actuate a small fraction of a system's states so that a specified control energy performance over the entire state space is guaranteed. In particular, we first consider the selection of a minimal number of actuated states so that an average control energy bound, along all the directions in the state space, is satisfied. Finding such a subset of states is a challenging task since it involves the search for a small number of actuators that induce controllability, which constitutes a combinatorial problem that can be computationally intensive. Indeed, identifying a small number of actuated states for inducing controllability alone is NP-hard [9]. Therefore, we extend this computationally hard problem by introducing an energy performance requirement on the choice of the optimal actuator set, and we solve it with an efficient approximation algorithm.

Specifically, we first generalize the involved energy objective to an ϵ-close one, which remains well defined even for actuator sets that render the system uncontrollable. Then, we make use of this metric and relax the implicit controllability constraint from the original actuator placement problem. Notwithstanding, we prove that for small values of ϵ, all solutions of this auxiliary program still render the system controllable. This fact, along with the supermodularity of the generalized objective with respect to the choice of the actuator set, leads to an efficient algorithm which, for a given range of problem parameters, approximates up to a multiplicative factor of $O(\log n)$, where n is the size of the system, any optimal actuator set that meets the specified energy criterion. Moreover, this is the best approximation factor that one can achieve in polynomial time, in the worst case. Hence, with this algorithm, we address the open problem of minimal actuator placement subject to bounds on the control effort [9], [11], [12], [14], [15].

Relevant results are also found in [12], where the authors study the controllability of a system with respect to the smallest eigenvalue of the controllability Gramian, and they derive a lower bound on the number of actuators so that this eigenvalue is lower bounded by a fixed value. Nonetheless, they do not provide an algorithm to identify the actuators that achieve this value. Our proposed algorithm, on the other hand, selects a minimal number of actuators so that this control objective is satisfied.

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Next, we consider the problem of cardinality-constrained actuator placement for minimum control effort, where the optimal actuator set is selected so that an average control energy objective around the entire state space is minimized. The most related works to this problem are [11] and [16], where the authors assume a controllable system and consider the problem of choosing a few extra actuators in order to optimize some of the input energy metrics proposed in [17]. Their main contribution is in observing that these energy metrics are supermodular with respect to the choice of the extra actuated states. The assumption of a controllable system is necessary since these metrics depend on the inverse of the controllability Gramian, since they quantify the average control energy for steering the system around the entire state space. Nonetheless, it should be also clear that making a system controllable by first placing some actuators to ensure controllability alone, and then adding some extra ones to optimize a desired energy metric, introduces a suboptimality that is carried over to the end result. In this paper, we follow a parallel line of work for the minimal actuator placement problem, and provide an efficient algorithm that selects all of the actuated states toward the minimization of an average control energy metric without any assumptions on the controllability of the involved system.

A similar actuator placement problem is studied in [12] for stable systems. Nevertheless, its authors propose a heuristic actuator placement procedure that does not constrain the number of available actuators and does not optimize their control energy objective. Our proposed algorithm selects a cardinality-constrained actuator set that minimizes a control energy metric, even for unstable systems.

The remainder of this paper is organized as follows. The formulation and model for the actuator placement problems are set forth in Section II, where the corresponding integer optimization programs are stated. In Sections III and IV, we discuss our main results, including the intractability of these problems, as well as the supermodularity of the involved control energy metrics with respect to the choice of the actuator sets. Then, we provide efficient approximation algorithms for their solution that guarantee a specified control energy performance over the entire state space. Finally, in Section V, we illustrate our analytical findings using an integrator chain network and test their efficiency over large random networks that are commonly used to model real-world networked systems. Our simulation results further stress the importance of selecting the actuators of a system for efficient control energy performance and not for mere controllability. Section VI concludes this paper.

II. Problem Formulation

Notation: We denote the set of natural numbers $\{1, 2, \ldots\}$ as $\mathbb{R}$, the set of real numbers as $\mathbb{R}$, and we let $[n] \equiv \{1, 2, \ldots, n\}$ for all $n \in \mathbb{R}$. Also, given a set $\mathcal{X}$, we denote as $|\mathcal{X}|$ its cardinality. Matrices are represented by capital letters and vectors by lower-case letters. For a matrix A, A^T is its transpose and A_{ij} is its element located at the ith row and jth column. If A is positive semidefinite or positive definite, we write $A \succeq 0$ and $A \succ 0$, respectively. Moreover, for $i \in [n]$, we let $I^{(i)}$ be an $n \times n$ matrix with a single nonzero element: $I_{ii} = 1$, while $I_{jk} = 0$, for j, $k \neq i$. Furthermore, we denote as I the identity matrix, whose dimension is inferred from the context. In addition, for $\delta \in \mathbb{R}^n$, we let $\text{diag}(\delta)$ denote an $n \times n$ diagonal matrix such that $\text{diag}(\delta)_{ii} = \delta_i$ for all $i \in [n]$. Finally, we set $\{0, 1\}^n$ to be the set of vectors in $\mathbb{R}^n$ whose elements are either zero or one.

A. Actuator Placement Model

Consider a linear system of n states, $x_1, x_2, \ldots, x_n$, whose evolution is described by

$$\dot{x}(t) = Ax(t) + Bu(t), t > t_0 \tag{1}$$

where $t_0 \in \mathbb{R}$ is fixed, $x \equiv \{x_1, x_2, \ldots, x_n\}$, $\dot{x}(t) \equiv dx/dt$, while u is the corresponding input vector. The matrices A and B are of appropriate dimension. We equivalently refer to (1) as a network of n nodes, $1, 2, \ldots, n$, which we associate with the states $x_1, x_2, \ldots, x_n$, respectively. Moreover, we denote their collection as $\mathcal{V} \equiv [n]$.

Henceforth, A is given while B is a *diagonal zero-one* matrix that we design so that (1) satisfies a specified control energy criterion over the entire state space.

Assumption 1: $B = \text{diag}(\delta)$, where $\delta \in \{0, 1\}^n$.

Specifically, if $\delta_i = 1$, state x_i may receive an input, while if $\delta_i = 0$, it receives none.

Definition 1 (Actuator Set, Actuator): Given a $\delta \in \{0, 1\}^n$, let $\Delta \equiv \{i : i \in \mathcal{V} \text{ and } \delta_i = 1\}$; then, Δ is called an *actuator set* and each $i \in \Delta$ an *actuator*.

B. Controllability and Related Average Energy Metrics

We consider the notion of controllability and relate it to the problems of this paper, that is, the minimal actuator placement for constrained control energy and the cardinality-constrained actuator placement for minimum control effort.

System (1) is controllable—equivalently, (A, B) is controllable—if for any finite $t_1 > t_0$ and any initial state $x_0 \equiv x(t_0)$ it can be steered to any other state $x_1 \equiv x(t_1)$ by some input $u(t)$ defined over $[t_0, t_1]$. Moreover, for general matrices A and B, the controllability condition is equivalent to the matrix

$$W \equiv \int_{t_0}^{t_1} \mathrm{e}^{A(t-t_0)} BB^T \mathrm{e}^{A^T(t-t_0)} \mathrm{d}t \tag{2}$$

being positive definite for any $t_1 > t_0$ [13]. Therefore, we refer to W as the *controllability matrix* of (1).

The controllability of a linear system is of interest because it is related to the solution of the following minimum-energy transfer problem

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \int_{t_0}^{t_1} u(t)^T u(t) \mathrm{d}t \\ \text{subject to} \quad & \dot{x}(t) = Ax(t) + Bu(t), t_0 < t \leq t_1 \\ & x(t_0) = x_0, x(t_1) = x_1 \end{aligned} \tag{3}$$

where A and B are any matrices of appropriate dimension.

In particular, if for the given A and B (1) is controllable, the resulting minimum control energy is given by

$$(x_1 - \mathrm{e}^{A\tau} x_0)^T W^{-1} (x_1 - \mathrm{e}^{A\tau} x_0) \tag{4}$$

where $\tau = t_1 - t_0$ [17]. Thereby, the states that belong to the eigenspace of the smallest eigenvalues of (2) require higher energies of control input [13]. Extending this observation along all the directions of transfers in the state space, we infer that the closer W is to singularity the larger the expected input energy required for these transfers to be achieved [17]. For example, consider the case where W is singular, that is, when there exists at least one direction along which system (1) cannot be steered [13]. Then, the corresponding minimum control energy along this direction is *infinity*.

This motivates the consideration of control energy metrics that quantify the steering energy along all the directions in the state space, as the $\mathrm{tr}(W^{-1})$ [17]; $\mathrm{tr}(W^{-1})$ measures the average minimum control effort over all transfers from the origin ($x_0 = 0$) to a point that is chosen uniformly from the unit sphere. Indeed, this metric is well defined only for controllable systems—W must be invertible—and is directly related to (4). In this paper, we aim to select a small number of actuators for system (1) so that $\mathrm{tr}(W^{-1})$ either meets a specified upper bound or is minimized.

Per Assumption 1, further properties for the controllability matrix are due: For any actuator set Δ, let $W_\Delta \equiv W$; then

$$W_\Delta = \sum_{i=1}^{n} \delta_i W_i \tag{5}$$

where $W_i \equiv \int_{t_0}^{t_1} \mathrm{e}^{At} I^{(i)} \mathrm{e}^{A^T t} \mathrm{d}t$ for any $i \in [n]$. This follows from (2) and the fact that $BB^T = B = \sum_{i=1}^{n} \delta_i I^{(i)}$ for $B = \mathrm{diag}(\delta)$. Finally, for any $\Delta_1 \subseteq \Delta_2 \subseteq \mathcal{V}$, (5) and $W_1, W_2, \ldots, W_n \succeq 0$ imply $W_{\Delta_1} \preceq W_{\Delta_2}$.

C. Actuator Placement Problems

We consider the selection of a small number of actuators for system (1) so that $\mathrm{tr}(W^{-1})$ either satisfies an upper bound or is minimized.[1] The challenge is in doing so with as few actuators as possible. This is an important improvement over the existing literature where the goal of actuator placement problems has either been to ensure controllability alone [9] or the weaker property of structural controllability [19], [20]. Other relevant results consider the task of leader-selection [21], [22], where the leaders are the actuated states and are chosen in order to minimize a mean-square convergence error of the remaining states.

Furthermore, the most relevant works to our study are the [11] and [16] since its authors consider the minimization of $\mathrm{tr}(W^{-1})$; nevertheless, their results rely on a pre-existing actuator set that renders (1) controllable although this set is not selected for the minimization of this energy metric. One of our contributions is in achieving optimal actuator placement for minimum average control effort without assuming controllability beforehand. Also, the authors of [12] adopt a similar framework for actuator placement but focus on deriving an upper bound for the smallest eigenvalue of W with respect to the number of actuators and a lower bound for the required number actuators so that this eigenvalue takes a specified value. In addition, they are motivated by the inequality $\mathrm{tr}(W^{-1}) \geq n^2/\mathrm{tr}(W)$, and consider the maximization of $\mathrm{tr}(W)$; however, their techniques cannot be applied when minimizing the $\mathrm{tr}(W^{-1})$, while the maximization of $\mathrm{tr}(W)$ may not ensure controllability [12].

We next provide the exact statements of our actuator placement problems, while their solution analysis follows in Sections III and IV. We first consider the problem

$$\begin{aligned} \underset{\Delta \subseteq \mathcal{V}}{\text{minimize}} \quad & |\Delta| \\ \text{subject to} \quad & \mathrm{tr}(W_\Delta^{-1}) \leq E \end{aligned} \tag{I}$$

for some constant E. Its domain is $\{\Delta : \Delta \subseteq \mathcal{V}$ and $(A, B(\Delta))$ is controllable$\}$ since the controllability matrix $W_{(\cdot)}$ must be invertible. Moreover, it is NP-hard, as we prove in Appendix A.

Additionally, Problem (I) is feasible for certain values of E. In particular, for any Δ such that $(A, B(\Delta))$ is controllable, $0 \prec W_\Delta$, that is, $\mathrm{tr}(W_{\mathcal{V}}^{-1}) \leq \mathrm{tr}(W_\Delta^{-1})$ since for any Δ (5) implies $W_\Delta \preceq W_{\mathcal{V}}$ [23]; thus, (I) is feasible for

$$E \geq \mathrm{tr}(W_{\mathcal{V}}^{-1}). \tag{6}$$

Moreover, (I) is a generalized version of the minimal controllability problem of [9] so that its solution not only ensures controllability but also satisfies a guarantee in terms of an average control energy metric; indeed, for $E \to \infty$, we recover the problem of [9].

We next consider the problem

$$\begin{aligned} \underset{\Delta \subseteq \mathcal{V}}{\text{minimize}} \quad & \mathrm{tr}(W_\Delta^{-1}) \\ \text{subject to} \quad & |\Delta| \leq r, \end{aligned} \tag{II}$$

where the goal is to find, at most, r actuated states so that an average control energy metric is minimized. Its domain is $\{\Delta : \Delta \subseteq \mathcal{V}, |\Delta| \leq r \text{ and } (A, B(\Delta)) \text{ is controllable}\}$. Moreover, due to the NP-hardness of Problem (I), Problem (II) is also NP-hard (cf. Appendix A).

Because (I) and (II) are NP-hard, we need to identify efficient approximation algorithms for their general solution; this is the subject of Sections III and IV. In particular, in Section III we consider Problem (I) and provide for it a best approximation algorithm, for a given range of problem parameters. To this end, we first define an auxiliary program, which ignores the controllability constraint of (I), and, nevertheless, admits an efficient approximation algorithm whose solutions not only satisfy an energy bound that is ϵ-close to the original one but also render system (1) controllable. Then, in Section IV we turn our attention to (II), and following a parallel line of thought as for (I), we efficiently solve this problem as well.

Since the approximation algorithm for the aforementioned auxiliary program for (I) is based on results for supermodular functions, we present below a brief overview of the relevant concepts. The reader may consult [24] for a survey on these results.

[1] In the companion paper [18], we extend these results to the transfer dependent case where the actuators are chosen so that the exact minimum required energy (4) for a given state transfer from x_0 to x_1 satisfies the same criteria.

D. Supermodular Functions

We give the definition of a supermodular function, as well as, a relevant result that will be used in Section III to construct an approximation algorithm for Problem (I). The material of this section is drawn from [25].

Let $\mathcal{V}$ be a finite set and denote as $2^{\mathcal{V}}$ its power set.

Definition 2 (Submodularity and Supermodularity): A function $h : 2^{\mathcal{V}} \mapsto \mathbb{R}$ is *submodular* if for any sets Δ and Δ', with $\Delta \subseteq \Delta' \subseteq \mathcal{V}$, and any $a \notin \Delta'$

$$h(\Delta \cup \{a\}) - h(\Delta) \geq h(\Delta' \cup \{a\}) - h(\Delta').$$

A function $h : 2^{\mathcal{V}} \mapsto \mathbb{R}$ is *supermodular* if $(-h)$ is submodular.

An alternative definition of a submodular function is based on the notion of nonincreasing set functions.

Definition 3 (Nonincreasing and Nondecreasing Set Function): A function $h : 2^{\mathcal{V}} \mapsto \mathbb{R}$ is a *nonincreasing set function* if for any $\Delta \subseteq \Delta' \subseteq \mathcal{V}$, $h(\Delta) \geq h(\Delta')$. Moreover, h is a *nondecreasing set function* if $(-h)$ is a nonincreasing set function.

Therefore, a function $h : 2^{\mathcal{V}} \mapsto \mathbb{R}$ is submodular if, for any $a \in \mathcal{V}$, the function $h_a : 2^{\mathcal{V}\setminus\{a\}} \mapsto \mathbb{R}$, defined as $h_a(\Delta) \equiv h(\Delta \cup \{a\}) - h(\Delta)$, is a nonincreasing set function. This property is also called the *diminishing returns property*.

Next, we present a fact from the supermodular functions minimization literature, that we use in Section III so as to construct an approximation algorithm for Problem (I). In particular, consider the following optimization program, which is of similar structure to (I), where $h : 2^{\mathcal{V}} \mapsto \mathbb{R}$ is a nondecreasing, supermodular set function

$$\begin{aligned} \underset{\Delta\subseteq\mathcal{V}}{\text{minimize}} \quad & |\Delta| \\ \text{subject to} \quad & h(\Delta) \leq E. \end{aligned} \qquad (\mathcal{O})$$

The following greedy algorithm has been proposed for its approximate solution, for which, the subsequent fact is true.

Algorithm 1 Approximation Algorithm for the Problem $(\mathcal{O})$.

Input: h, E.
Output: Approximate solution to Problem $(\mathcal{O})$.
$\Delta \leftarrow \emptyset$
while $h(\Delta) > E$ **do**
$a_i \leftarrow a' \in \arg\max_{a\in\mathcal{V}\setminus\Delta}\{h(\Delta) - h(\Delta \cup \{a\})\}$
$\Delta \leftarrow \Delta \cup \{a_i\}$
end while

Fact 1: Denote $\Delta^\star$ as a solution to Problem $(\mathcal{O})$ and as $\Delta_0, \Delta_1, \ldots$ the sequence of sets picked by Algorithm 1. Moreover, let l be the smallest index such that $h(\Delta_l) \leq E$. Then

$$\frac{l}{|\Delta^\star|} \leq 1 + \log\frac{h(\mathcal{V}) - h(\emptyset)}{h(\mathcal{V}) - h(\Delta_{l-1})}.$$

In Section III, we provide an efficient approximation algorithm for (I), by applying Fact 1 to an appropriately perturbed version of this problem, so that it involves a nondecreasing supermodular function, as in $(\mathcal{O})$. This also leads to our second main contribution, presented in Section IV: An efficient approximation algorithm for Problem (II), which selects all the actuators towards minimum average control effort, without assuming controllability beforehand. This is in contrast to the related works [11] and [16]: there, the authors consider a similar problem for choosing a few actuators to optimize $\mathrm{tr}(W_{(\cdot)}{}^{-1})$; however, their results rely on the assumption of a pre-existing actuator set that renders (1) controllable, although this set is not selected toward the minimization of $\mathrm{tr}(W_{(\cdot)}{}^{-1})$. Nevertheless, this assumption is necessary, since they then prove that the $\mathrm{tr}(W_{(\cdot)}{}^{-1})$ is a supermodular function in the choice of the extra actuators. On the other hand, our algorithms select all the actuators towards the involved energy objective, since they rely on a ϵ-perturbed version of $\mathrm{tr}(W_{(\cdot)}{}^{-1})$, that we prove to be supermodular without assuming controllability beforehand.

Another related work to supermodularity and ours is [26], where the authors consider the problem of choosing a few actuators to maximize $\mathrm{tr}(W_{(\cdot)})$, motivated by the inequality $\mathrm{tr}(W^{-1}) \geq n^2/\mathrm{tr}(W)$. Their main contribution is in proving that this metric is modular[2] in the choice of the actuators. However, their techniques cannot be applied when minimizing the $\mathrm{tr}(W^{-1})$, while the maximization of $\mathrm{tr}(W)$ may not ensure controllability [12].

Overall, our results supplement the existing literature by considering Problems (I) and (II) when the system is not initially controllable and by providing efficient approximation algorithms for their solution, along with worst-case performance guarantees.

III. Minimal Actuator Sets With Constrained Average Control Effort

We present an efficient approximation algorithm for Problem (I). To this end, we first generalize the involved energy metric to an ϵ-close one that remains well defined even when the controllability matrix is not invertible. Next, we relax (I) by introducing a new program that makes use of this metric and circumvents the restrictive controllability constraint of (I). Moreover, we prove that for certain values of ϵ all solutions of this auxiliary problem render the system controllable. This fact, along with the supermodularity property of the generalized metric that we establish, leads to our proposed approximation algorithm. The discussion of its efficiency ends the analysis of (I).

A. ϵ-Close Auxiliary Problem

Consider the following approximation to (I)

$$\begin{aligned} \underset{\Delta\subseteq\mathcal{V}}{\text{minimize}} \quad & |\Delta| \\ \text{subject to} \quad & \mathrm{tr}(W_\Delta + \epsilon I)^{-1} \leq E \end{aligned} \qquad (\mathrm{I}')$$

where ϵ satisfies $0 < \epsilon \leq 1/E$. In contrast to (I), the domain of this problem consists of all subsets of $\mathcal{V}$ since $W_{(\cdot)} + \epsilon I$ is always invertible.

[2] A set function is modular if it is both submodular and supermodular.

The ϵ-closeness is evident since for any Δ such that $(A, B(\Delta))$ is controllable $\mathrm{tr}(W_\Delta+\epsilon I)^{-1} \rightarrow \mathrm{tr}(W_\Delta^{-1})$ as $\epsilon \rightarrow 0$. Notice that we can take $\epsilon \rightarrow 0$ since we assume any positive $\epsilon \leq 1/E$.

In the following paragraphs, we identify an approximation algorithm for solving Problem (I′), and correspondingly, the ϵ-close, NP-hard Problem (I).

B. Approximation Algorithm for Problem (I′)

We first prove that all solutions of (I′) for $0 < \epsilon \leq 1/E$ render the system controllable, notwithstanding that no controllability constraint is imposed by this program on the choice of the actuator sets. Moreover, we show that the involved ϵ-close energy metric is supermodular with respect to the choice of actuator sets and then we present our approximation algorithm, followed by a discussion of its efficiency which ends this subsection.

Proposition 1: Consider a constant $\omega > 0$, ϵ such that $0 < \epsilon \leq 1/\omega$, and any $\Delta \subseteq \mathcal{V}$: If $\mathrm{tr}(W_\Delta + \epsilon I)^{-1} \leq \omega$, then $(A, B(\Delta))$ is controllable.

Proof: Assume that $(A, B(\Delta))$ is not controllable and let k be the corresponding number of nonzero eigenvalues of W_Δ which we denote as $\lambda_1, \lambda_2, \ldots, \lambda_k$; therefore, $k \leq n-1$. Then

$$\mathrm{tr}(W_\Delta + \epsilon I)^{-1} = \sum_{i=1}^{k} \frac{1}{\lambda_i + \epsilon} + \frac{n-k}{\epsilon} > \frac{1}{\epsilon} \geq \omega$$

and since $\epsilon \leq 1/\omega$, we have a contradiction. □

Note that ω is chosen independently of the parameters of system (1). Therefore, the absence of the controllability constraint in Problem (I′) for $0 < \epsilon \leq 1/E$ is fictitious; nonetheless, it obviates the necessity of considering only actuator sets that render the system controllable.

The next proposition is also essential and suggests an efficient approximation algorithm for solving (I′).

Proposition 2 (Supermodularity): The function $\mathrm{tr}(W_\Delta + \epsilon I)^{-1} : \Delta \subseteq \mathcal{V} \mapsto \mathbb{R}$ is supermodular with respect to the choice of Δ.

Proof: From Section II-D, recall that $\mathrm{tr}(W_\Delta + \epsilon I)^{-1}$ is supermodular if and only if $-\mathrm{tr}(W_\Delta + \epsilon I)^{-1}$ is submodular, as well as that a function $h : 2^{\mathcal{V}} \mapsto \mathbb{R}$ is submodular if and only if, for any $a \in \mathcal{V}$, the function $h_a : 2^{\mathcal{V}\setminus\{a\}} \mapsto \mathbb{R}$, where $h_a(\Delta) \equiv h(\Delta \cup \{a\}) - h(\Delta)$ is a nonincreasing set function. Therefore, to prove that $\mathrm{tr}(W_\Delta + \epsilon I)^{-1}$ is supermodular, we may prove that the $h_a(\Delta) = -\mathrm{tr}(W_{\Delta\cup\{a\}} + \epsilon I)^{-1} + \mathrm{tr}(W_\Delta + \epsilon I)^{-1}$ is a nonincreasing set function:

For any $a \in \mathcal{V}$, take any $\Delta_1 \subseteq \Delta_2 \subseteq \mathcal{V} \setminus \{a\}$ and denote accordingly $\mathcal{D} \equiv \Delta_2 \setminus \Delta_1$; thus, we aim to prove that

$$-\mathrm{tr}(W_{\Delta_1\cup\{a\}} + \epsilon I)^{-1} + \mathrm{tr}(W_{\Delta_1} + \epsilon I)^{-1} \geq -\mathrm{tr}(W_{\Delta_1\cup\mathcal{D}\cup\{a\}} + \epsilon I)^{-1} + \mathrm{tr}(W_{\Delta_1\cup\mathcal{D}} + \epsilon I)^{-1}.$$

To this end, and for $z \in [0, 1]$, set $f(z) \equiv \mathrm{tr}(W_{\Delta_1} + zW_{\mathcal{D}} + Wa + \epsilon I)^{-1}$ and $g(z) \equiv \mathrm{tr}(W_{\Delta_1} + zW_{\mathcal{D}} + \epsilon I)^{-1}$. After some manipulations, the above inequality is written as $f(1) - f(0) \geq g(1) - g(0)$.

Therefore, to prove this one instead, it suffices to prove that $df/dz \geq dg/dz$ for any $z \in (0, 1)$: Denote $L_1(z) \equiv W_{\Delta_1} + zW_{\mathcal{D}} + W_a + \epsilon I$ and $L_2(z) \equiv W_{\Delta_1} + zW_{\mathcal{D}} + \epsilon I$. Then, the $df/dz \geq dg/dz$ becomes

$$\mathrm{tr}(L_1(z)^{-1} W_{\mathcal{D}} L_1(z)^{-1}) \leq \mathrm{tr}(L_2(z)^{-1} W_{\mathcal{D}} L_2(z)^{-1}) \quad (7)$$

where we used the fact that for any $A \succ 0$, $B \succeq 0$ and $z \in (0, 1)$, $(d/dz)\mathrm{tr}((A + zB)^{-1}) = -\mathrm{tr}((A + zB)^{-1} B (A + zB)^{-1})$.

Now, to show that (7) holds, first observe that $L_1(z) \succeq L_2(z)$. This implies $L_2(z)^{-1} \succeq L_1(z)^{-1}$ and, as a result, $L_2(z)^{-2} \succeq L_1(z)^{-2}$ [23]. Hence, $W_{\mathcal{D}}^{1/2} L_2(z)^{-2} W_{\mathcal{D}}^{1/2} \succeq W_{\mathcal{D}}^{1/2} L_1(z)^{-2} W_{\mathcal{D}}^{1/2}$, which gives

$$\mathrm{tr}(W_{\mathcal{D}}^{1/2} L_2(z)^{-2} W_{\mathcal{D}}^{1/2}) \geq \mathrm{tr}(W_{\mathcal{D}}^{1/2} L_1(z)^{-2} W_{\mathcal{D}}^{1/2}).$$

Finally, the cycle property of trace yields inequality (7). Consequently, $\mathrm{tr}(W_\Delta + \epsilon I)^{-1}$ is supermodular. □

Therefore, the hardness of the ϵ-close Problem (I) is in agreement with that of the class of minimum set-covering problems subject to submodular constraints. Inspired by this literature [24], [25], [27], we have the following efficient approximation algorithm for Problem (I′), and as we show by the end of this section, for Problem (I) as well.

Algorithm 2 Approximation Algorithm for the Problem I′.

Input: Bound E, parameter $\epsilon \leq 1/E$, matrices $W_1, W_2, \ldots, W_n$.
Output: Actuator set Δ.
$\Delta \leftarrow \emptyset$
while $\mathrm{tr}(W_\Delta + \epsilon I)^{-1} > E$ **do**
$a_i \leftarrow a' \in \arg\max_{a \in \mathcal{V}\setminus\Delta} \{\mathrm{tr}(W_\Delta + \epsilon I)^{-1} - \mathrm{tr}(W_{\Delta\cup\{a\}} + \epsilon I)^{-1}\}$
$\Delta \leftarrow \Delta \cup \{a_i\}$
end while

Regarding the quality of Algorithm 2 the following is true.

Theorem 1 (A Submodular Set Coverage Optimization): Denote as $\Delta^\star$ a solution to Problem (I′) and as Δ the selected set by Algorithm 2. Then

$$(A, B(\Delta)) \text{ is controllable} \quad (8)$$

$$\mathrm{tr}(W_\Delta + \epsilon I)^{-1} \leq E \quad (9)$$

$$\frac{|\Delta|}{|\Delta^\star|} \leq 1 + \log \frac{n\epsilon^{-1} - \mathrm{tr}(W_{\mathcal{V}} + \epsilon I)^{-1}}{E - \mathrm{tr}(W_{\mathcal{V}} + \epsilon I)^{-1}} \equiv F \quad (10)$$

$$F = O(\log n + \log \epsilon^{-1} + \log \frac{1}{E - \mathrm{tr}(W_{\mathcal{V}}^{-1})}). \quad (11)$$

Finally, the computational complexity of Algorithm 2 is $O(n^5)$.

Proof: We first prove (9), (10) and (11), and then, (8). We end the proof by clarifying the computational complexity of Algorithm 2.

First, let $\Delta_0, \Delta_1, \ldots$ be the sequence of sets selected by Algorithm 2 and l the smallest index such that

$\mathrm{tr}(W_{\Delta_l}+\epsilon I)^{-1} \leq E$. Therefore, Δ_l is the set that Algorithm 2 returns, and this proves (9).

Moreover, from [25], since for any $\Delta \subseteq \mathcal{V}$, $h(\Delta) \equiv -tr(W_\Delta+\epsilon I)^{-1}+n\epsilon^{-1}$ is a non-negative, nondecreasing, and submodular function (cf. Proposition 2), it is guaranteed for Algorithm 2 that (cf. Fact 1)

$$\frac{l}{|\Delta^\star|} \leq 1+\log\frac{h(\mathcal{V})-h(\emptyset)}{h(\mathcal{V})-h(\Delta_{l-1})}$$
$$=1+\log\frac{n\epsilon^{-1}-\mathrm{tr}(W_\mathcal{V}+\epsilon I)^{-1}}{\mathrm{tr}(W_{\Delta_{l-1}}+\epsilon I)^{-1}-\mathrm{tr}(W_\mathcal{V}+\epsilon I)^{-1}}.$$

Now, l is the first time that $\mathrm{tr}(W_{\Delta_l}+\epsilon I)^{-1} \leq E$, and a result $\mathrm{tr}(W_{\Delta_{l-1}}+\epsilon I)^{-1} > E$. This implies (10).

Moreover, observe that $0<\mathrm{tr}(W_\mathcal{V}+\epsilon I)^{-1}<\mathrm{tr}(W_\mathcal{V}^{-1})$ so that from (10) we get $F \leq 1+\log[n\epsilon^{-1}/(E-\mathrm{tr}(W_\mathcal{V}^{-1}))]$ which, in turn, implies (11).

On the other hand, since $0<\epsilon\leq 1/E$ and $\mathrm{tr}(W_{\Delta_l}+\epsilon I)^{-1} \leq E$, Proposition 1 is in effect, that is, (8) holds true.

Finally, with respect to the computational complexity of Algorithm 2, note that the `while` loop is repeated for, at most, n times. Moreover, the complexity to invert an $n\times n$ matrix, using Gauss-Jordan elimination decomposition, is $O(n^3)$. In addition, at most, n matrices must be inverted so that the "$\arg\max_{a\in\mathcal{V}\setminus\Delta}\{\mathrm{tr}(W\Delta+\epsilon I)^{-1}-\mathrm{tr}(W_{\Delta\cup\{a\}}+\epsilon I)^{-1}\}$" can be computed. Furthermore, $O(n)$ time is required to find a maximum element between n available. Therefore, the computational complexity of Algorithm 2 is $O(n^5)$. □

Therefore, Algorithm 2 returns a set of actuators that meets the corresponding control energy bound of Problem (I′) while it renders system (1) controllable. Moreover, the cardinality of this set is up to a multiplicative factor of F from the minimum cardinality actuator sets that meet the same control energy bound.

The dependence of F on n, ϵ and E was expected from a design perspective: Increasing the network size n or improving the accuracy by decreasing ϵ, as well as demanding a better energy guarantee by decreasing E should all push the cardinality of the selected actuator set upwards. Also, $\log\epsilon^{-1}$ is the design cost for circumventing the difficult to satisfy controllability constraint of (I) [9], that is, for assuming no preexisting actuators that renders (1) controllable and choosing all the actuators towards the satisfaction of an energy performance criterion.

From a computational perspective, the matrix inversion is the only intensive procedure of Algorithm 2, requiring $O(n^3)$ time, if we use the Gauss-Jordan elimination decomposition. On the other hand, to apply this algorithm on large-scale systems, we can speed up this procedure using the Coppersmith-Winograd algorithm [28], which requires $O(n^{2.376})$ time. Alternatively, we can use numerical methods, which efficiently compute an approximate inverse of a matrix even if its size is of several thousands [29], [30]. Moreover, we can speed up Algorithm 2 using a method proposed in [31], which avoids the computation of $\mathrm{tr}(W_\Delta+\epsilon I)^{-1}-\mathrm{tr}(W_{\Delta\cup\{a\}}+\epsilon I)^{-1}$ for unnecessary choices of a, towards the computation of the $\arg\max_{a\in\mathcal{V}\setminus\Delta}\{\mathrm{tr}(W_\Delta+\epsilon I)^{-1}-\mathrm{tr}(W_{\Delta\cup\{a\}}+\epsilon I)^{-1}\}$, by taking advantage of the supermodularity of $\mathrm{tr}(W_{(\cdot)}+\epsilon I)^{-1}$.

Finally, for large values of n, the computation of $W_1, W_2, \ldots, W_n$ is demanding as well. On the other hand, in the case of stable systems, as many physical (for example, biological) networks are, the corresponding controllability Gramians can be used instead which, for a stable system, can be calculated from the Lyapunov equations $AG_i+G_iA^T=-I^{(i)}$, for $i=1,2,\ldots,n$, respectively, and are given in closed-form by

$$G_i=\int_{t_0}^{\infty} \mathrm{e}^{A(t-t_0)}I^{(i)}\mathrm{e}^{A^T(t-t_0)}\,\mathrm{d}t. \tag{12}$$

Using these Gramians for the evaluation of W in (4) corresponds to the minimum state transfer energy with no time constraints. The advantage of this approach is that (12) can be solved efficiently using numerical methods, even when the system's size n has a value of several thousands [32].

In Section III-C we finalize our treatment of Problem (I) by employing Algorithm 2 to approximate its solutions.

C. Approximation Algorithm for Problem (I)

We present an efficient approximation algorithm for Problem (I) that is based on Algorithm 2. Let Δ be the actuator set returned by Algorithm 2, so that $(A, B(\Delta))$ is controllable and $\mathrm{tr}(W_\Delta+\epsilon I)^{-1}\leq E$. Moreover, denote as $\lambda_1(W_\Delta), \lambda_2(W_\Delta), \ldots, \lambda_n(W_\Delta)$ the eigenvalues of W_Δ and as $\lambda_m(W_\Delta)$ the smallest one. Finally, consider a positive ϵ such that $n\epsilon/\lambda_m^2(W_\Delta)\leq cE$ for some $c>0$. Then

$$\mathrm{tr}(W_\Delta+\epsilon I)^{-1}=\sum_{i=1}^{n}\frac{1}{\lambda_i(W_\Delta)+\epsilon}\geq \tag{13}$$

$$=\mathrm{tr}(W_\Delta^{-1})-\sum_{i=1}^{n}\frac{\epsilon}{\lambda_i^2(W_\Delta)} \tag{14}$$

$$=\mathrm{tr}(W_\Delta^{-1})-\frac{n\epsilon}{\lambda_m^2(W_\Delta)}$$

$$\geq \mathrm{tr}(W_\Delta^{-1})-cE \tag{15}$$

where we derived (14) from (13) using that for any $x\geq 0$, $1/(1+x)\geq 1-x$, while the rest follow from the definition of $\lambda_m(W_\Delta)$ and the assumption $n\epsilon/\lambda_m^2(W_\Delta)\leq cE$. Moreover, $\mathrm{tr}(W_\Delta+\epsilon I)^{-1}\leq E$, and therefore we get from (15) that

$$\mathrm{tr}(W_\Delta^{-1})\leq(1+c)E. \tag{16}$$

Hence, we refer to c as *approximation error*.

On the other hand, $\lambda_m(W_\Delta)$ is not known a priori. Hence, we need to search for a sufficiently small ϵ so that (16) holds true. One way to achieve this since ϵ is lower and upper bounded by 0 and $1/E$, respectively, is to perform a search using bisection. We implement this procedure in Algorithm 3, where we denote as [Algorithm 2](E,ϵ) the set that Algorithm 2 returns for given E and ϵ.

Algorithm 3 Approximation Algorithm for the Problem (I).

Input: Bound E, approximation error c, bisection's initial accuracy level a_0, matrices $W_1, W_2, \ldots, W_n$.
Output: Actuator set Δ.

```
a ← a_0, flag ← 0, l ← 0, u ← 1/E, ε ← (l + u)/2
while flag ≠ 1 do
  while u − l > a do
  Δ ← [Algorithm 2](E, ε)
    if tr(W_Δ^{-1}) − tr(W_Δ + εI)^{-1} > cE then
      u ← ε
    else
      l ← ε
    end if
  ε ← (l + u)/2
  end while
  if tr(W_Δ^{-1}) − tr(W_Δ + εI)^{-1} > cE then
    u ← ε, ε ← (l + u)/2
  end if
  Δ ← [Algorithm 2](E, ε)
  if tr(W_Δ^{-1}) − tr(W_Δ + εI)^{-1} ≤ cE then
    flag ← 1
  else
    a ← a/2
  end if
end while
```

In the worst case, when we first enter the inner `while` loop, the `if` condition is not satisfied and, as a result, ϵ is set to a lower value. This process continues until the `if` condition is satisfied for the first time, given that a_0 is sufficiently small for the specified c, from which point and on this `while` loop converges up to the accuracy level a_0 to the largest value $\bar{\epsilon}$ of ϵ such that $\mathrm{tr}(W_\Delta^{-1}) - \mathrm{tr}(W_\Delta + \epsilon I)^{-1} \leq cE$; specifically, $|\epsilon - \bar{\epsilon}| \leq a_0/2$, due to the mechanics of the bisection method. On the other hand, if a_0 is not sufficiently small, the value of a decreases within the last `if` statement of the algorithm, the variable flag remains zero and the outer loop is executed again, until the convergence within the inner `while` is feasible. Then, the `if` statement that follows the inner `while` loop ensures that ϵ is set below $\bar{\epsilon}$, so that $\mathrm{tr}(W_\Delta^{-1}) - \mathrm{tr}(W_\Delta + \epsilon I)^{-1} \leq cE$. Finally, the last `if` statement sets the flag to 1 and the algorithm terminates. The efficiency of this algorithm for Problem (I) is summarized below.

Theorem 2 (Approximation Efficiency and Computational Complexity of Algorithm 3 for Problem (I)): Denote as $\Delta^\star$ a solution to Problem (I) and as Δ the selected set by Algorithm 3. Then

$$(A, B(\Delta)) \text{ is controllable}$$

$$\mathrm{tr}(W_\Delta^{-1}) \leq (1 + c)E \quad (17)$$

$$\frac{|\Delta|}{|\Delta^\star|} \leq F \quad (18)$$

$$F = O\left(\log n + \log \frac{E}{c} + \log \frac{1}{E - \mathrm{tr}(W_\mathcal{V}^{-1})}\right). \quad (19)$$

Finally, let a be the bisection's accuracy level that Algorithm 3 terminates with. Then, if $a = a_0$, the computational complexity of Algorithm 3 is $O(n^5 \log_2(1/(a_0 E))$, else it is $O(n^5 \log_2(1/(aE)) \log_2(a_0/a))$.

Proof: We only prove statements (17), (18) and (19), while the first follows from Theorem 1. We end the proof by clarifying the computational complexity of Algorithm 3.

First, when Algorithm 3 exits the `while` loop, and after the following `if` statement, $\mathrm{tr}(W_\Delta^{-1}) - \mathrm{tr}(W_\Delta + \epsilon I)^{-1} \leq cE$, and since $\mathrm{tr}(W_\Delta + \epsilon I)^{-1} \leq E$, this implies (17).

To show (18), consider any solution $\Delta^\star$ to Problem (I) and any solution $\Delta^\bullet$ to Problem (I′). Then, $|\Delta^\star| \geq |\Delta^\bullet|$; to see this, note that for any $\Delta^\star$, $\mathrm{tr}(W_{\Delta^\star} + \epsilon I)^{-1} < \mathrm{tr}(W_{\Delta^\star}^{-1}) \leq E$ since $\epsilon > 0$, that is, $\Delta^\star$ is a candidate solution to Problem (I′) because it satisfies all of its constraints. Therefore, $|\Delta^\star| \geq |\Delta^\bullet|$ and as a result $|\Delta|/|\Delta^\star| \leq |\Delta|/|\Delta^\bullet| \leq F$ per (10).

Next, note that (17) holds true when ϵ is of the same order as $n/(c\lambda_m^2(W_\Delta)E)$, while $1/\lambda_m(W_\Delta) < \mathrm{tr}(W_\Delta^{-1}) = O(E)$. Therefore, $\log \epsilon^{-1} = O(\log n + \log E/c)$ and this proves (19).

Finally, with respect to the computational complexity of Algorithm 3, note that the inner `while` loop is repeated for, at most, $\log_2(1/(aE))$ times, in the worst case. Moreover, the time complexity of the procedures within this loop is of order $O(n^5)$, due to Algorithm 2. Finally, if $a = a_0$, the outer `while` loop runs for one time, and otherwise, for $\log_2(a_0/a)$ times. Therefore, the computational complexity of Algorithm 3 is $O(n^5 \log_2(1/(a_0 E)))$, or $O(n^5 \log_2(1/(aE)) \log_2(a_0/a))$, respectively. ■

From a computational perspective, we can speed up Algorithm 3 using the methods we discussed in the end of Section III-B. Moreover, for a wide class of systems, for example, when $E = O(n^{n^{c_1}})$, where c_1 is a positive constant, independent of n, and similarly for a, this algorithm runs in polynomial time, due to the logarithmic dependence on E and a, respectively.

From an approximation efficiency perspective we have that $F = O(\log(n))$, whenever $E = O(n^{c_1})$ and $1/(E - \mathrm{tr}(W_\mathcal{V}^{-1})) = O(n^{c_2})$, where c_1 and c_2 are positive constants and independent of n. In other words, the cardinality of the actuator set that Algorithm 3 returns is up to a multiplicative factor of $O(\log n)$ from the minimum cardinality actuator sets that meet the same energy bound. Indeed, this is the best achievable bound in polynomial time for the set covering problem in the worst case [33], while (I) is a generalization of it [9]. Thus, Algorithm 3 is a best-approximation of (I) for this class of systems.

IV. Minimum Energy Control by a Cardinality-Constrained Actuator Set

We present an approximation algorithm for Problem (II) following a parallel line of thought as in Section III: First, we circumvent the restrictive controllability constraint of (II) using the ϵ-close generalized energy metric defined in Section III. Then, we propose an efficient approximation algorithm for its solution that makes use of Algorithm 3; this algorithm returns an actuator set that always renders (1) controllable while it guarantees a value for (II) that is provably close to its optimal one. We end the analysis of (II) by explicating further the efficiency of this procedure.

A. ϵ-Close Auxiliary Problem

For $\epsilon > 0$, consider the following approximation to (II):

$$\begin{aligned} \underset{\Delta \subseteq \mathcal{V}}{\text{minimize}} \quad & \text{tr}(W_\Delta + \epsilon I)^{-1} \\ \text{subject to} \quad & |\Delta| \leq r. \end{aligned} \tag{II$'$}$$

In contrast to (II), the domain of this problem consists of all subsets of $\mathcal{V}$ since $W_{(\cdot)} + \epsilon I$ is always invertible. Moreover, its objective is ϵ-close to that of Problem (II).

In the following paragraphs, we identify an efficient approximation algorithm for solving Problem (II$'$), and correspondingly, the ϵ-close, NP-hard Problem (II). We note that the hardness of the latter is in accordance with that of the general class of supermodular function minimization problems, as per Proposition 2 the objective $\text{tr}(W_\Delta + \epsilon I)^{-1}$ is supermodular. The approximation algorithms used in that literature, however, [24], [25], and [27] fail to provide an efficient solution algorithm for (II$'$)—for completeness, we discuss this direction in Appendix B. In the next subsection, we propose an efficient approximation algorithm for (II) that makes use of Algorithm 3.

B. Approximation Algorithm for Problem (II)

We provide an efficient approximation algorithm for Problem (II) that is based on Algorithm 3. In particular, since (II) finds an actuator set that minimizes $\text{tr}(W_{(\cdot)}^{-1})$, and any solution to (I) satisfies $\text{tr}(W_{(\cdot)}^{-1}) \leq E$, one may repeatedly execute Algorithm 3 for decreasing values of E as long as the returned actuators are, at most, r and E satisfies the feasibility constraint $E \geq \text{tr}(W_{\mathcal{V}}^{-1})$ (cf. Section II-C). Therefore, for solving (II) we propose a bisection-type execution of Algorithm 3 with respect to E.

To this end, we also need an upper bound for the value of (II): Let $\Delta_{\mathcal{C}}$ be a small actuator set that renders system (1) controllable; it is efficiently found using Algorithm 3 for large E or the procedure proposed in [9]. Then, for any $r \geq |\Delta_{\mathcal{C}}|$, $\text{tr}(W_{\Delta_{\mathcal{C}}}^{-1})$ upper bounds the value of (II) since $\text{tr}(W_{(\cdot)}^{-1})$ is monotone.

Thus, having a lower and upper bound for the value of (II), we implement Algorithm 4 for approximating the solutions of (II); we consider only the nontrivial case where $r < n$ and denote the set that Algorithm 3 returns as $[\text{Algorithm 3}](E, c, a_0)$ for given E, c, and a_0.

Algorithm 4 Approximation algorithm for Problem (II).

Input: Set $\Delta_{\mathcal{C}}$, maximum number of actuators r such that $r \geq |\Delta_{\mathcal{C}}|$, approximation error c for Algorithm 3, bisection's accuracy level a_0 for Algorithm 3, bisection's accuracy level a_0' for current algorithm, matrices $W_1, W_2, \ldots, W_n$.
Output: Actuator set Δ.

$\Delta \leftarrow \emptyset,\ l \leftarrow \text{tr}(W_{\mathcal{V}}^{-1}),\ u \leftarrow \text{tr}(W_{\Delta_{\mathcal{C}}}^{-1}),\ E \leftarrow (l+u)/2,$ $\epsilon \leftarrow 1/E$
while $u - l > a_0'$ **do**
$\Delta \leftarrow [\text{Algorithm 3}](E, c, a_0)$
  if $|\Delta| > r$ **then**
    $l \leftarrow E, E \leftarrow (l+u)/2$
  else
    $u \leftarrow E, E \leftarrow (l+u)/2$
  end if
$\epsilon \leftarrow 1/E$
end while
if $|\Delta| > r$ **then**
  $l \leftarrow E, E \leftarrow (l+u)/2$
end if
$\Delta \leftarrow [\text{Algorithm 3}](E, c, a_0)$

In the worst case, when we first enter the `while` loop, the `if` condition is not satisfied and, as a result, E is set to a greater value. This process continues until the `if` condition is satisfied for the first time from which point and on the algorithm converges up to the accuracy level a_0 to the smallest value $\underline{E}$ of E such that $|\Delta| \leq r$; specifically, $|E - \underline{E}| \leq a_0'/2$ due to the mechanics of the bisection method, where $\underline{E} \equiv \min\{E : |[\text{Algorithm 3}](E, c, a_0)| \leq r\}$. Hereby $\underline{E}$ is the least bound E for which Algorithm 3 returns an actuator set of cardinality, at most, r for the specified c and a_0—$\underline{E}$ may be larger than the value of (II) due to worst-case approximability of the involved problems (cf. Theorem 2). Then, Algorithm 4 exits the `while` loop and the last `if` statement ensures that E is set below $\underline{E}$ so that $|\Delta| \leq r$. Moreover, per Theorem 2 this set renders (1) controllable and guarantees that $\text{tr}(W_\Delta^{-1}) \leq (1+c)E$. Finally, with respect to the computational complexity of Algorithm 4, note that the `while` loop is repeated for, at most, $\log_2[(\text{tr}(W_{\Delta_{\mathcal{C}}}^{-1}) - \text{tr}(W_{\mathcal{V}}^{-1}))/a_0']$ times. Moreover, the time complexity of the procedures within this loop are, in the worst case, of the same order as that of Algorithm 3 when it is executed for E equal to $\underline{E}$. Regarding Theorem 2, denote this time complexity as $C(\underline{E}, c, a_0)$. Therefore, the computational complexity of Algorithm 3 is $O(C(\underline{E}, c, a_0)\log_2[(\text{tr}(W_{\Delta_{\mathcal{C}}}^{-1}) - \text{tr}(W_{\mathcal{V}}^{-1}))/a_0'])$.

We summarize the above in the next corollary, which also ends the analysis of Problem (II).

Corollary 1 (Approximation Efficiency and Computational Complexity of Algorithm 4 for Problem (II)): Denote as Δ the selected set by Algorithm 4. Then

$$(A, B(\Delta)) \text{ is controllable}$$
$$\text{tr}(W_\Delta^{-1}) \leq (1+c)E$$
$$|E - \underline{E}| \leq a_0'/2$$

where $\underline{E} = \min\{E : |[\text{Algorithm 3}](E, c, a_0)| \leq r\}$ is the least bound E that Algorithm 3 satisfies with an actuator set of cardinality, at most, r for the specified c and a_0. Finally, the computational complexity of Algorithm 4 is

$$O\left(C(\underline{E}, c, a_0)\log_2\left(\frac{\text{tr}(W_{\Delta_{\mathcal{C}}}^{-1}) - \text{tr}(W_{\mathcal{V}}^{-1})}{a_0'}\right)\right)$$

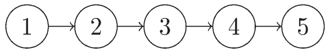

Fig. 1. Five-node integrator chain.

where $C(\underline{E}, c, a_0)$ denotes the computational complexity of Algorithm 3, with respect to Theorem 2, when it is executed for E equal to $\underline{E}$.

From a computational perspective, we can speed up Algorithm 4 using the methods we discussed at the end of Section III-B. Moreover, for a wide class of systems, for example, when $E = O(n^{n^{c_1}})$, where c_1 is a positive constant, independent of n, and similarly for a_0, a_0' and $\mathrm{tr}(W_{\Delta_C}^{-1})$, this algorithm runs in polynomial time, due to the logarithmic dependence on E, a_0, a_0', and $\mathrm{tr}(W_{\Delta_C}^{-1})$, respectively.

V. Examples and Discussions

We test the performance of Algorithms 3 and 4 over various systems starting with the case of an integrator chain network in Section V-A and following up with the Erdös–Rényi random networks in Section V-B.

A. Integrator Chain Network

We illustrate the mechanics and efficiency of Algorithms 3 and 4 using the integrator chain in Fig. 1, where

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

We first run Algorithm 3 for $t_0 = 0, t_1 = 1, E \leftarrow \mathrm{tr}(W_{\{1,5\}}^{-1})$ and a_0, $c \leftarrow .0001$. The algorithm returned the actuator set $\{1, 3\}$; as expected, node 1 is chosen since, for a chain network to be controllable, 1 must be an actuator. Moreover, $\{1, 3\}$ is the best actuator set, as it follows by comparing the values below that were computed using MATLAB:

$$\mathrm{tr}(W_{\{1\}}^{-1}) = 8.5175 \cdot 10^7$$
$$\mathrm{tr}(W_{\{1,2\}}^{-1}) = 3.3234 \cdot 10^5$$
$$\mathrm{tr}(W_{\{1,3\}}^{-1}) = 2.4209 \cdot 10^3$$
$$\mathrm{tr}(W_{\{1,4\}}^{-1}) = 2.4221 \cdot 10^3$$
$$\mathrm{tr}(W_{\{1,5\}}^{-1}) = 3.3594 \cdot 10^5.$$

Therefore, node 1 does not satisfy by itself the bound E, while $\mathrm{tr}(W_{\{1,3\}}^{-1})$ not only satisfies this bound but also takes the smallest value among all the actuator sets of cardinality two that induce controllability; therefore, $\{1, 3\}$ is the best minimal actuator set to achieve the given transfer.

Furthermore, observe from these values that the average control energy by the minimum actuator set that induces controllability alone, that is, $\{1\}$, is of order four times larger than the best actuator set that includes just one more node, which is $\{1, 3\}$. This illustrates that even for the simple network of Fig. 1, it is important to select its actuators for efficient control energy performance and not controllability alone.

Finally, by setting E large enough in Algorithm 3 so that the bound $\mathrm{tr}(W_{(\cdot)}^{-1}) \leq E$ is satisfied by any set in the domain of (I), we observe that only node 1 is selected, as expected for the controllability constraint of this domain to be met.

We next run Algorithm 4 for $t_0 = 0$, $t_1 = 1$, a_0, a_0', $c \leftarrow .0001$, and r being equal to 1, 2, 3, 4 or 5, respectively; node 1 is always selected, while for every value of r, the chosen actuator set coincides with the one that has the same size and minimizes $\mathrm{tr}(W_{(\cdot)}^{-1})$. That is, we again observe optimal performance by our algorithm.

Furthermore, by increasing r from 1 to 3, the corresponding value of $\mathrm{tr}(W_{(\cdot)}^{-1})$ decreases from $85.1750 \cdot 10^6$ to 81.7134, a difference of six orders. Therefore, we notice again that for a system to be efficiently controllable its actuators must be chosen for bounded control effort and not controllability alone.

B. Erdös–Rényi Random Networks

Erdös–Rényi random graphs are commonly used to model real-world networked systems [1]. According to this model, each edge is included in the generated graph with some probability p independently of every other edge. We implemented this model for varying network sizes n where the directed edge probabilities were set to $p = 2\log(n)/n$, following [9]. In particular, we first generated the binary adjacencies matrices for each network size so that each edge is present with probability p and then we replaced every nonzero entry with an independent standard normal variable to generate a randomly weighted graph.

To avoid the computational difficulties associated with the integral equation (2), we worked with the controllability Gramian instead, which for a stable system can be efficiently calculated from the Lyapunov equation $AG + GA^T = -BB^T$ and is given in closed-form by

$$G = \int_{t_0}^{\infty} \mathrm{e}^{A(t-t_0)} BB^T \mathrm{e}^{A^T(t-t_0)} \, \mathrm{d}t.$$

Using the controllability Gramian in (4) corresponds to the minimum state transfer energy with no time constraints. Therefore, we stabilized each random instances of A by subtracting 1.1 times the real part of their right-most eigenvalue and then we used the MATLAB function `gram` to compute the corresponding controllability Gramians.

Next, we set $c \leftarrow 0.1$ and $a_0 \leftarrow 1$. Then, we run Algorithm 3 for n equal to 10, 40, 70, and 100, respectively, and E equal to $k \cdot \mathrm{tr}(G(n)_{\mathcal{V}}^{-1})$, where k ranged from 2 to 2^{50}; $\mathrm{tr}(G(n)_{\mathcal{V}}^{-1})$ is the lower bound of E so that (I) is feasible for each n (cf. (6)). The corresponding number of actuators that Algorithm 3 selected with respect to k is shown in Fig. 2, where the horizontal axis is in logarithmic scale. Finally, we also set $a_0' \leftarrow 1$, and run Algorithm 4 for n equal to 10, 40, 70, and 100, respectively (for each generated network of size n, we first run Algorithm 3 for $E \leftarrow 10^{18} \cdot \mathrm{tr}(G(n)_{\mathcal{V}}^{-1})$, and c, a_0 as before, and we found a $\Delta_{\mathcal{C}}(n)$ such that $|\Delta_{\mathcal{C}}(n)| = 1$); the corresponding achieved values for Problem (II) with respect to r are found in Fig. 3, where the vertical axis is in logarithmic scale.

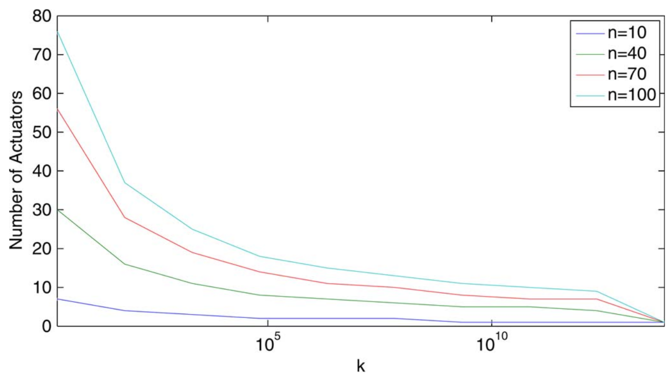

Fig. 2. Number of selected actuators by Algorithm 3 in Erdös–Rényi random networks of size n and for varying energy bounds E (the horizontal axis is in logarithmic scale); for each n the values of E are chosen so that the feasibility constraint (6) of Problem (I) is satisfied: Specifically, for each n and k, Algorithm 3 is executed for $E \leftarrow k \cdot \mathrm{tr}(G(n)_{\mathcal{V}}^{-1})$, where $G(n)_{\mathcal{V}}$ is the controllability Gramian corresponding to the generated network of size n when all of its nodes are actuated.

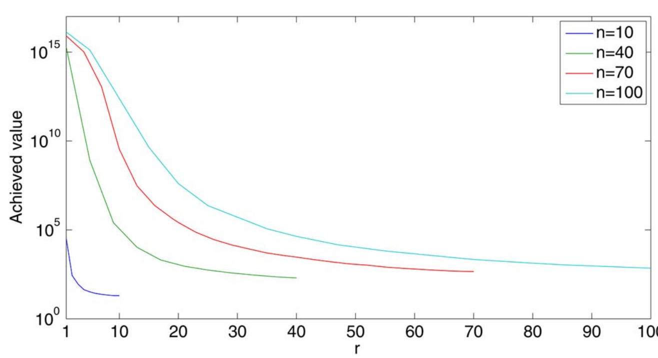

Fig. 3. Minimum-achieved value for $\mathrm{tr}(W_{(\cdot)}^{-1})$ by Algorithm 4 in Erdös–Rényi random networks of size n and for varying number of available actuators r (the vertical axis is in logarithmic scale).

In Fig. 2, we observe that as k increases the number of actuators decreases, as one would expect when the energy bound of (I) is relaxed. Moreover, for k large enough, so that (I) becomes equivalent to the minimal controllability problem of [9], the number of chosen actuators is one; that is, Algorithm 3 outperforms the theoretical guarantees of Theorem 2. Similarly, in Fig. 3 we observe that as the number of available actuators r increases the minimum achieved value also decreases, as expected by the monotonicity and supermodularity of $\mathrm{tr}(W_{(\cdot)}^{-1})$, while as r decreases, the minimum-achieved value blows up.

Hence, Figs. 2 and 3 complement the results of Section V-A: For large E, all of the networks are controllable from one node, while as E decreases, that is, when the constraint on the control effort becomes more strict, the number of necessary actuators increases; similarly, as the number of available actuators r decreases, the minimum control effort increases and, in fact, almost doubly exponentially fast. Moreover, for each n, the value of $\mathrm{tr}(W_{\Delta_{\mathcal{C}}(n)}^{-1})$, which is the control energy associated with a single actuator selected merely for controllability, is of four orders larger than the corresponding optimized value of $\mathrm{tr}(W_{(\cdot)}^{-1})$ in Fig. 3 for $r = 1$.

Thereby, our observations on both the integrator chain and the Erdös–Rényi networks support that a system may be practically uncontrollable if its actuators are chosen so that only the theoretical controllability conditions for the system are satisfied; thus, it is important to incorporate in the choice of these actuators the associated control energy performance, as we undertook in this paper.

VI. Concluding Remarks

We addressed two actuator placement problems in a linear system: First, there is the problem of minimal actuator placement so that an average control energy bound along all the directions in the state space is satisfied, and then there is the problem of cardinality-constrained actuator placement for minimum average control effort. Both problems were shown to be NP-hard, while for the first one, we provided a best approximation algorithm for a given range of the problem parameters. Next, we proposed an efficient approximation algorithm for the solution of the second problem as well. Finally, we illustrated our analytical findings using an integrator chain network and demonstrated their efficiency over large Erdös–Rényi random networks. Our future work is focused on exploring the effect that the underlying network topology of the involved system has on these actuator placement problems as well as investigating distributed implementations of their corresponding algorithms.

Appendix

A. Computational Complexity of Problems (I) and (II)

We prove that Problem I is NP-hard, providing an instance that reduces to the NP-hard controllability problem introduced in [9]. In particular, it is shown in [9] that deciding if (1) is controllable by a zero-one diagonal matrix B with $r+1$ nonzero entries reduces to the r-hitting set problem, as we define it below, which is NP-hard [34].

Definition 4 (r-Hitting Set Problem): Given a finite set $\mathcal{M}$ and a collection $\mathcal{C}$ of nonempty subsets of $\mathcal{M}$, find an $\mathcal{M}' \subseteq \mathcal{M}$ of cardinality, at most, r that has a nonempty intersection with each set in $\mathcal{C}$.

Without loss of generality, we assume that every element of $\mathcal{M}$ appears in at least one set in $\mathcal{C}$ and all sets in $\mathcal{C}$ are nonempty. Moreover in Definition 4, we let $|\mathcal{C}| = p$ and $\mathcal{M} = \{1, 2, \ldots, m\}$, and define $C \in \mathbb{R}^{p \times m}$ such that $C_{ij} = 1$ if the ith set contains the element j and zero otherwise.

Theorem 3 (Computational Complexity of Problem (I)): Problem (I) is NP-hard.

Proof: We show that Problem (I) for A as described below and with $E = n(2n)^{2n^2+12n+2}$ is equivalent to the NP-hard controllability problem introduced in [9]. Therefore, since E can be described in polynomial time, as $log(E) = O(n^3)$, we conclude that Problem (I) is NP-hard.

In particular, as in [9], let $n = m + p + 1$ and $A = V^{-1}DV$, where $D \equiv \mathrm{diag}(1, 2, \ldots, n)$ and[3]

$$V = \begin{bmatrix} 2I_{m\times m} & 0_{m\times p} & e_{m\times 1} \\ C & (m+1)I_{p\times p} & 0_{p\times 1} \\ 0_{1\times m} & 0_{1\times p} & 1 \end{bmatrix}. \quad (20)$$

[3] V is invertible since it is strictly diagonally dominant.

It is shown in [9] that deciding if A is controllable by a zero-one diagonal matrix B with $r+1$ nonzero entries is NP-hard.

Now, observe that all the entries of V are integers either zero or, at most, $m+1$. Moreover, with respect to the entries of V^{-1}, it is shown in [9] that:

- For $i = 1, 2, \ldots, m$: It has a $1/2$ in the (i,i)th place and a $-1/2$ in the (i,n)th place, and zeros elsewhere.
- For $i = m+1, m+2, \ldots, m+p$: It has a $1/(m+1)$ in the (i,i)th place, a $-1/(2(m+1))$ in the (i,j)th place where $j \in C_i$ (C_i is the corresponding set of the collection $\mathcal{C}$), and $|C_i|/(2(m+1))$ in the (i,n)th place; every other entry of the ith row is zero.
- Finally, the last row of V^{-1} is $[0, 0, \ldots, 0, 1]$.

Therefore, $2(m+1)V^{-1}$ has all its entries as integers that are either zero or, at most, n^2, in absolute value.

Consider the controllability matrix associated with this system, given a zero-one diagonal B that makes it controllable, and denote it as W_B. Then

$$W_B = \int_{t_0}^{t_1} \mathrm{e}^{A(t-t_0)} B B^T \mathrm{e}^{A^T(t-t_0)}\, \mathrm{d}t$$

$$= V^{-1} \int_{t_0}^{t_1} \mathrm{e}^{D(t-t_0)} V B V^T \mathrm{e}^{D^T(t-t_0)}\, \mathrm{d}t V^{-T}.$$

Let $t_1 - t_0 = ln(n)$. Then$(2n)! \int_0^{t_1-t_0} \mathrm{e}^{Dt} V B V^T \mathrm{e}^{D^T t} \mathrm{d}t$ evaluates to a matrix that has entries of the form $c_0 + c_1 n + c_2 n^2 + \ldots + c_n n^n$, where $c_0, c_1, \ldots, c_n$ are non-negative integers and all less than $(2n)! \leq (2n)^{2n}$. Thereby

$$W'_B \equiv 4(m+1)^2 (2n)! V^{-1} \int_0^{t_1-t_0} \mathrm{e}^{Dt} V B V^T \mathrm{e}^{D^T t} \mathrm{d}t V^{-T}$$

has entries of the form $c'_0 + c'_1 n + c'_2 n^2 + \ldots + c'_n n^n$, where $c'_0, c'_1, \ldots, c'_n$ are integers and all less than $(2n)^{2(n+3)}$ in absolute value due to the pre and post multiplications by $2(m+1)V^{-1}$ and $2(m+1)V^{-T}$, respectively.

We are interested on upper bounding $tr(W_B^{-1}) = 4(m+1)^2(2n)! tr({W'_B}^{-1}) \leq (2n)^{2(n+1)} tr({W'_B}^{-1})$. Therefore, we upper bound $tr({W'_B}^{-1})$: Using Crammer's rule to compute ${W'_B}^{-1}$, due to the form of the entries of W'_B, all of its elements, including the diagonal ones, if they approach infinity, they approach it with, at most, $n! n^n (2n)^{2n(n+3)} < (2n)^{2n(n+5)}$ speed and, as a result, $tr({W'_B}^{-1}) \leq n(2n)^{2n(n+5)}$. Hence, $tr(W_B^{-1}) \leq n(2n)^{2n(n+5)+2(n+1)} = n(2n)^{2n^2+12an+2}$, for any B that makes (1) controllable. Thus, if we set $E = n(2n)^{2n^2+12n+2}$ (which implies $log(E) = O(n^3)$ so that E can be described polynomially), Problem (I) is equivalent to the controllability problem of [9], which is NP-hard. □

An immediate consequence of Theorem 3 is the following one.

Corollary 2 (Computational Complexity of Problem (II)): Problem (II) is NP-hard.

B. Greedy Algorithm Used in the Supermodular Minimization Literature is Inefficient for Solving Problem (II′)

Consider Algorithm 5, which is in accordance with the supermodular minimization literature [24], [25], [27].

Algorithm 5 Greedy Algorithm for Problem (II′).

Input: Maximum number of actuators r, approximation pa rameter ϵ, number of steps that the algorithm will run l, matrices $W_1, W_2, \ldots, W_n$.
Output: Actuator set Δ_l
$\Delta_0 \leftarrow \emptyset$, $i \leftarrow 0$
while $i < l$ **do**
$a_i \leftarrow \arg\max_{a \in \mathcal{V} \setminus \Delta} \{\mathrm{tr}(W_{\Delta_i} + \epsilon I)^{-1} - \mathrm{tr}(W_{\Delta_i \cup \{a\}} + \epsilon I)^{-1}\}$
$\Delta_{i+1} \leftarrow \Delta_i \cup \{a_i\}, i \leftarrow i+1$
end while

The following is true for its performance.

Fact 2: Let $v^\star$ denote the value of Problem (II′). Then, Algorithm 5 guarantees that for any positive integer l

$$\mathrm{tr}(W_{\Delta_l} + \epsilon I)^{-1} \leq (1 - e^{-l/r}) v^\star + \frac{n e^{-l/r}}{\epsilon}.$$

Proof: It follows from [27, Theorem 9.3, Ch. III.3.9.], since $-\mathrm{tr}(W_\Delta + \epsilon I)^{-1} + n\epsilon^{-1}$ is a non-negative, nondecreasing, and submodular function with respect to the choice of Δ (cf. Proposition 2). □

Therefore, Algorithm 5 suffers from an error term that is proportional to ϵ^{-1}. On the other hand, if we set $l = 5r$, then we reduce the error term $ne^{-l/r}/\epsilon$ from $0.37n/\epsilon$ to $0.01n/\epsilon$; however, this is achieved at the expense of violating the cardinality constraint. Moreover, it is possible that Algorithm 5 returns an actuator set that does not render (1) controllable. Therefore, Algorithm 5 is inefficient for solving Problem (II′).

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