Reactive and Risk-Aware Control for Signal Temporal Logic

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Abstract—The deployment of autonomous systems in uncertain and dynamic environments has raised fundamental questions. Addressing these is pivotal to build fully autonomous systems and requires a systematic integration of planning and control. We first propose reactive risk signal interval temporal logic (ReRiSITL) as an extension of signal temporal logic (STL) to formulate complex spatiotemporal specifications. Unlike STL, ReRiSITL allows to consider uncontrollable propositions that may model humans as well as random environmental events such as sensor failures. Additionally, ReRiSITL allows to incorporate risk measures, such as (but not limited to) the conditional value-at-risk, to measure the risk of violating certain spatial specifications. Second, we propose an algorithm to check if an ReRiSITL specification is satisfiable. For this purpose, we abstract the ReRiSITL specification into a timed signal transducer and devise a game-based approach. Third, we propose a reactive planning and control framework for dynamical control systems under ReRiSITL specifications.

Index Terms—Formal methods, reactive control, risk-aware control, signal temporal logic (STL), timed automata.

I. INTRODUCTION

TEMPORAL logics allow to express temporal properties in a logical framework providing an expressive specification language. Signal temporal logic (STL) is a predicate-based temporal logic that offers many appealing advantages [1]. In particular, STL allows to impose quantitative temporal properties, e.g., combinations of surveillance (“visit regions A, B, and C every 10–60 s”), safety (“always between 5–25 s stay at least 1 m away from D”), and many others. Indeed, there is a rich body of literature on the control of dynamical systems under STL specifications, e.g., [2]–[4].

However, a key obstacle to deploying such control frameworks in real-world settings is to account for uncertain and dynamic environments. In particular, objects of interests may be estimated by simultaneous localization and mapping algorithms and be described as probability distributions, see e.g., [5] and [6], so that one may want to consider risk. Also, random events such as sensor failures or humans requesting assistance play an increasing role. While there has been recent work addressing some of these challenges, e.g., [7]–[10], there exists no reactive and risk-aware planning and control framework with formal correctness guarantees. We claim that no one has rigorously addressed the reactive planning problem for systems under STL specifications. Toward addressing this shortcoming, we leverage ideas from formal methods, risk theory, control theory, game theory, and timed automata.

Related Work: For the control under STL specifications, mixed integer linear programs [2], [11], [12] have been presented that encode the STL specification. Nonconvex optimization programs [3], [13] and reinforcement learning approaches [14], [15] have further been proposed and particularly use the quantitative semantics associated with an STL specification [16]. A timed automata-based planning framework has been presented in our previous work [17], where we decompose the STL specification into STL subspecifications. Feedback control laws that implement such STL subspecifications, which are timed transitions, have appeared in [4], [18]–[24].

Linear temporal logic (LTL) is a proposition-based temporal logic, less expressive than STL, that allows to impose qualitative temporal properties. Existing control approaches leverage automata-based synthesis [25]–[27]. Metric interval temporal logic (MITL) is a proposition-based temporal logic with quantitative temporal properties [28], hence more expressive than LTL but less expressive than STL. An MITL specification can be translated into a language equivalent timed automaton [28]. If the accepted language of this automaton is not empty [29], the MITL specification is satisfiable. For pointwise MITL semantics, a tool to perform this translation has been presented in [30]. Pointwise semantics, however, do not guarantee the satisfaction of the MITL specification in continuous time. The procedure of [28], for continuous-time semantics, is complex and not compositional. The results from [31], [32] are more intuitive and present a compositional way to control timed automata by reformulating it as a timed two player game, played between controllable (the system) and uncontrollable (environment) events, see also [34]–[36].
The underlying assumption in these previous works is that the environment is perfectly known. For LTL, this assumption has been relaxed in [5], [6], and [37]. Specifically, the authors in [5] and [6] assumed that the environment is modeled as a semantic map. Target beliefs in surveillance games and Markov decision process-based approaches are presented in [38] and [39]. Probabilistic computational tree logic and distribution temporal logic [40] account for state distributions and can take chance constraints into account, but only consider qualitative temporal properties and do not consider risk measures [41], [42]. The works in [43] and [44] consider the generalized reactivity (1) fragment, which explicitly accounts for dynamic environments. For STL, the works in [7] and [9] consider chance constraints, whereas [8] and [45] already incorporate risk measures without, however, considering random environmental events. Such events have been considered for STL in [10]. The proposed reactive control strategy in [10] has been evaluated empirically, but without providing formal guarantees. A reactive counter-example-guided framework was proposed in [46] where, however, the risk of violating certain spatial specifications is not considered. Furthermore, only bounded specifications are considered while the STL specification is not allowed to explicitly depend on the environment.

Contributions: In this article, our first contribution is to propose reactive risk signal interval temporal logic (ReRiSITL). Compared with STL, ReRiSITL has two distinct features and hence generalizes STL. First, ReRiSITL specifications may contain uncontrollable propositions that allow to model humans, or in general other agents, and environmental events such as sensor failures or communication dropouts. Second, ReRiSITL allows to incorporate risk measures by considering risk predicates so that the risk of violating certain spatial specifications can be taken into account. Such risk predicates can take different risk measures into account, as for instance the conditional value-at-risk (CVaR). Our second contribution is an algorithm that allows to check if such an ReRiSITL specification is satisfiable. To do so, we abstract the ReRiSITL specification into a timed signal transducer using and adapting the results from [32] and then follow a game-based strategy similarly to [33]. The third contribution is a planning and control framework for dynamical control systems under ReRiSITL specifications. The main elements here are a well-defined timed abstraction of the control system that relies on existing feedback control laws as presented in [18]–[24]. We then propose to use a combination of a game-based approach, graph search techniques, and replanning. We remark that our approach is, to the best of our knowledge, the first to incorporate past temporal operators and we hereby establish a connection between monitoring and reactive control.

Structure: Section II presents ReRiSITL and the problem formulation. Section III presents the algorithm to check if an ReRiSITL specification is satisfiable. Sections IV and V propose the planning and control framework for dynamical control systems under ReRiSITL specifications. Simulations and conclusions are provided in Sections VII and VIII.

II. PRELIMINARIES AND PROBLEM FORMULATION

True and false are encoded as $\top := \infty$ and $\bot := -\infty$ with $\mathbb{B} := \{\top, \bot\}$; $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ are the real, rational, and natural numbers, respectively, while $\mathbb{R}_{\geq 0}$ and $\mathbb{Q}_{\geq 0}$ denote their respective nonnegative (positive) subsets. For $t \in \mathbb{R}_{\geq 0}$ and $I \subseteq \mathbb{R}_{\geq 0}$, let $t \cap I$ and $t \cup I$ denote the Minkowski sum and the Minkowski difference of $t$ and $I$, respectively. For two sets $\mathcal{X}$ and $\mathcal{Y}$, we use the notation $\mathcal{F}(\mathcal{X}, \mathcal{Y})$ to denote the set of all measurable functions that map from $\mathcal{X}$ to $\mathcal{Y}$. An element $f \in \mathcal{F}(\mathcal{X}, \mathcal{Y})$ is hence a function $f : \mathcal{X} \rightarrow \mathcal{Y}$. Let $(\Omega, \mathcal{B}_1, \mathcal{P}_1)$ be a probability space, where $\mathcal{X}$ is the sample space, $\mathcal{B}_1$ is the Borel $\sigma$-algebra of $\Omega$, and $\mathcal{P}_1 : \mathcal{B}_1 \rightarrow [0, 1]$ is a probability measure. A vector of random variables is a measurable function $\mathcal{X} : \Omega \rightarrow \mathbb{R}^n$ defined on a probability space $(\Omega, \mathcal{B}_1, \mathcal{P}_1)$, where $n \in \mathbb{N}$. We can associate the probability space $(\mathbb{R}^n, \mathcal{B}_\mathbb{R}^n, \mathcal{P}_X)$ with $\mathcal{X}$, where for $B \in \mathcal{B}_\mathbb{R}^n$, $P_X(B) := \mathcal{P}_1(X^{-1}(B))$ and $X^{-1}(B) := \{\omega \in \Omega | X(\omega) \in B\}$. Let $\tilde{\mu} := \mathcal{E}[X]$, and $\Sigma$ be the expected value and covariance matrix of $\mathcal{X}$, respectively, while $\mathcal{N}(\tilde{\mu}, \Sigma)$ denotes the multivariate normal distribution. We remark that all important symbols that have been or will be used in this article are summarized in Table I.

| Table I: SUMMARY OF THE MOST IMPORTANT NOTATION USED THROUGHOUT THE ARTICLE |
|------------------------|------------------------|
| Symbol | Meaning |
| $\mathcal{F}(\mathcal{X}, \mathcal{Y})$ | Set of all measurable functions mapping from a set $\mathcal{X}$ into a set $\mathcal{Y}$ |
| $x, s$ | The function $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ denotes a deterministic signal, while the element $s \in \mathcal{F}(\mathbb{R}_{\geq 0}, \mathbb{B}^{\mathbb{M}_n})$ denotes a random signal. |
| $\mathcal{X} : \tilde{\mu}, \Sigma$ | The function $\mathcal{X} : \Omega \rightarrow \mathbb{R}^n$ denotes a random variable with expected value $\tilde{\mu} \in \mathbb{R}^n$ and covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$. |
| $h$ | The function $h : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes predicate functions. |
| $\mathcal{M}^R, \mathcal{M}^M, \mathcal{M}^M$ | $\mathcal{M}^R$ set of risk predicates, $\mathcal{M}^M$ set of uncontrollable propositions, $\mathcal{M}$ set of risk predicates and uncontrollable propositions. |
| $\mu^{\mathbb{R}_{\geq 0}} = \mu^n \in \mathcal{M}^R$ | The element $\mu^n \in \mathcal{M}^R$ is a risk predicate, while the element $\mu^n \in \mathcal{M}^M$ is an uncontrollable proposition. |
| $\mathcal{F} : (\mathbb{F}, \mathcal{B}) \rightarrow \mathbb{R}$ | The functions $\mathcal{F} : (\mathbb{F}, \mathcal{B}) \rightarrow \mathbb{R}$ denotes a risk measure, $\beta$ is a risk level, and $\gamma$ is a risk threshold. |
| $\mathcal{A}^T$ | Set of (atomic) propositions for MITL specifications. |
| $\mathcal{R} \rightarrow \mathcal{Y}$ | The function $\mathcal{R} \rightarrow \mathcal{Y}$ denotes the set of all Boolean combinations over $\mathcal{A}$. |
| $\mathcal{T} \sigma$ | The transformation $\sigma$ transforms an ReRiSITL specification $\phi$ into an MITL specification $\sigma \phi$; $\mathcal{T}^{-1}$ is the inverse. |
| $\mathcal{T}^{\mathcal{S} \sigma}$ | Timed signal transducers for the MITL specification $\phi$ and the ReRiSITL specification $\phi$. |
| $\mathcal{R} \mathcal{A} \mathcal{C}, \mathcal{R} \mathcal{A} \mathcal{C}$ | The functions $\mathcal{R} \mathcal{A} \mathcal{C}$, $\mathcal{R} \mathcal{A} \mathcal{C}$ are different versions of the region automation of $\mathcal{T}^{\mathcal{S} \sigma}$. |
| $\mathcal{d}_p, d_{p}$ | The plan $d_p : \mathbb{R}_{\geq 0} \rightarrow \mathcal{B}(\mathcal{A})$ is constructed for a specification $\phi$, $d_{p}$ is simply its projection to $\mathcal{M}$ via $\mathcal{T}^{-1}$. |
| $\mathcal{X}^{m}, \mathcal{X}^{m_{\sigma}}, \mathcal{X}^{m_{\mathcal{F}_n}}, X^{m_{\mathcal{F}_n}}$ | The sets $\mathcal{X}^{m}, \mathcal{X}^{m_{\sigma}}, \mathcal{X}^{m_{\mathcal{F}_n}}, X^{m_{\mathcal{F}_n}}$ are risk constrained sets that are determined into the set $\mathcal{X}_m$. |
| $\mu^{\mathbb{R}_{\geq 0}} | \mathcal{M}^R$ | The element $\mu^{\mathbb{R}_{\geq 0}} | \mathcal{M}^R$ is a deterministic predicate, $\mathcal{M}$ is the set of deterministic predicates and uncontrollable propositions. |
| $\mathcal{T}^{\mathcal{S} \sigma}$ | Timed signal transducers for the ReRiSITL specification $\mathcal{S}$ and the product automaton. |
space \((\Omega, \mathcal{B}, P)\). At time \(t\), the probability space \((\mathbb{R}, \mathcal{B}, P_h)\) can be associated with \(h(x(t), X)\), a random variable, where \(P_h\) is derived from the probability space \((\mathbb{R}^n, \mathcal{B}^n, P_X)\).

We consider risk predicates for ReRiSITL based on risk measures as advocated in [41] and [42] toward an axiomatic risk assessment. A risk measure \(R : \mathcal{F}(\Omega, \mathbb{R}) \rightarrow \mathbb{R}\) allows to exclude behavior which is deemed more risky than other behavior. We are interested in \(R(-h(x(t), X))\) to argue about the risk of violating \(h(x(t), X) \geq 0\). The truth value of a risk predicate \(\mu^{R_i} : \mathbb{R}^n \times \mathbb{B} \rightarrow \mathbb{B}\) at time \(t\) is obtained as

\[
\mu^{R_i}(x(t), X) := \begin{cases} 
\top & \text{if } R(-h(x(t), X)) \leq \gamma \\
\bot & \text{otherwise}
\end{cases}
\]

for a risk threshold \(\gamma \in \mathbb{R}\). There are various choices of \(R(\cdot)\), see [42] for an overview. We consider the expected value (EV), the VaR, and the CVaR. The expected value of \(-h(x(t), X)\), denoted by \(EV[-h(x(t), X)]\), provides a risk neutral risk measure. More risk averse are the VaR and the CVaR as in [41]. The VaR of \(-h(x(t), X)\) for \(\beta \in (0, 1)\) is defined as

\[
\text{VaR}_\beta(-h(x(t), X)) := \min\{d \in \mathbb{R} \mid P_h(-h(x(t), X) \leq d) \geq \beta\}
\]

i.e., the worst case \(1 - \beta\) probability quantile.

Remark 1: Note that \(\text{VaR}_\beta(-h(x(t), X)) \leq \gamma\) is equivalent to \(P_h(-h(x(t), X) \leq \gamma) \geq \beta\) so that our framework includes chance constraints as for instance used in [9]. The CVaR of \(-h(x(t), X)\) for a risk level \(\beta\) is given by

\[
\text{CVaR}_\beta(-h(x(t), X)) := EV[-h(x(t), X)]
\]

i.e., the conditional expected value of \(-h(x(t), X)\) relative to \(-h(x(t), X)\) being greater than or equal to the VaR. Let now \(M^{R_i}\) denote a set of risk predicates.

Let \(M^{uc}\) be a set of uncontrollable propositions \(\mu^{uc}\) and \(s \in \mathcal{F}(\mathbb{R}^{\mathbb{Q}_0}, \mathbb{B}^{M^{R_i}})\) be a random Boolean signal corresponding to the truth values of the propositions in \(M^{R_i}\) over time. Define also the projection of \(s\) onto \(\mu^{uc} \in M^{M^{R_i}}\) as \(\text{proj}_{\mu^{uc}}(s) : \mathbb{R}_0 \rightarrow \mathbb{B}\), i.e., the truth value of \(\mu^{uc}\) over time.

Let \(\mu \in M := M^{R_i} \cup M^{uc}\) be a risk predicate or an uncontrollable proposition. The syntax of ReRiSITL is

\[
\phi := \top \mid \mu \mid \neg \phi \mid \phi \land \phi' \mid \phi U_t \phi'' \mid \phi U_t \phi''
\]

where \(\phi'\) and \(\phi''\) are ReRiSITL formulas and where \(U_t\) and \(U_{t'}\) are the future and past until operators. We restrict the time interval \(I\) to belong to the nonnegative rationals, i.e., \(I \subseteq \mathbb{Q}_0\). Additionally, we require that \(I\) is not a singleton, i.e., \(I\) is not allowed to be of the form \(I := [a, a]\) for \(a \in \mathbb{Q}_0\). Note that the former assumption is not restrictive, while the latter excludes punctuality constraints. We remark that these assumptions are commonly made [28]. Also define \(\phi' \lor \phi'' := (\neg (\neg \phi') \land \neg \phi'')\) (disjunction), \(F_t \phi := U_t \phi\) (future eventually), \(F_{t'} \phi := U_{t'} \phi\) (past eventually), \(G_t \phi := \neg F_t \neg \phi\) (future always), \(G_{t'} \phi := \neg F_{t'} \neg \phi\) (past always). We say that an ReRiSITL formula \(\phi\) is in positive normal form if no negation occurs within

1. We remark that \(X\) can be assumed to be a stochastic process \(X(t)\). To avoid further technical complexity, this is not followed in this article.

2. The proposition \(\mu^{uc}\) is labeled uncontrollable because \(s\) is assumed to be a random signal generated by an unknown underlying stochastic process, as highlighted by the notation \(s \in \mathcal{F}(\mathbb{R}^{\mathbb{Q}_0}, \mathbb{B}^{M^{R_i}})\).

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Fig. 1. Overview of the workspace in Example 1.

\[\phi[11].\text{ Let } (x, s, X, t) \models \phi\text{ denote the satisfaction relation as defined next.}\]

Definition 1 (ReRiSITL Semantics): We recursively define the continuous-time semantics of ReRiSITL as

\[
(x, s, X, t) \models \mu^{R_i} \quad \text{iff} \quad R(-h(x(t), X)) \leq \gamma
\]

\[
(x, s, X, t) \models \mu^{uc} \quad \text{iff} \quad \text{proj}_{\mu^{uc}}(s)(t) = T
\]

\[
(x, s, X, t) \models \neg \phi \quad \text{iff} \quad \neg((x, s, X, t) \models \phi)
\]

\[
(x, s, X, t) \models \phi' \land \phi'' \quad \text{iff} \quad (x, s, X, t) \models \phi' \land (x, s, X, t) \models \phi''
\]

\[
(x, s, X, t) \models \phi' U_t \phi'' \quad \text{iff} \quad \exists t'' \in t \oplus I, \text{ such that } (x, s, X, t') \models \phi', (x, s, X, t'') \models \phi''
\]

\[
(x, s, X, t) \models \phi' U_{t'} \phi'' \quad \text{iff} \quad \exists t' \in t \ominus I, \text{ such that } (x, s, X, t') \models \phi', (x, s, X, t'') \models \phi''
\]
where $\epsilon := 0.5$ and $R_{R1}(\cdot)$ and $R_{R2}(\cdot)$ encode the VaR with $\beta_{R1} = \beta_{R2} := 0.8$ and $\gamma_{R1} = \gamma_{R2} := 0$. Recall that, according to Remark 1, the risk predicate $R_{R1}$ using VaR encodes the probability that $h_{R1}^\phi(x(t), X) \geq 0$ is greater than 0.8. Let also

$$h_{R1}^\phi(x(t), X) := \|x(t) - X_{O1}\|^2 - \epsilon$$

where the risk measures $R_{O1}(\cdot)$ and $R_{O2}(\cdot)$ encode the CVaR with $\beta_{O1} = \beta_{O2} := 0.9$ and $\gamma_{O1} = \gamma_{O2} := 0$. To define satisﬁability of an ReRiSITL speciﬁcation, we need to take into account that propositions in $M^{sc}$ are uncontrollable. We ﬁrst deﬁne what a nonanticipative strategy is. A strategy $x_{na} : \mathcal{F}(\mathbb{R}_{\geq 0}, [0,1]) \to \mathcal{F}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ is nonanticipative if: For any $t \geq 0$ and for any two signals $s, s' \in \mathcal{F}(\mathbb{R}_{\geq 0}, [0,1])$ with $s(\tau) = s'(\tau)$ for all $\tau \in [0, t]$, it holds that $x_{na}(s)(\tau) = x_{na}(s')(\tau)$ for all $\tau \in [0, t]$. This means that $x_{na}(s)$ takes, at time $t$, only current and past values of $s$ into account, i.e., $s(\tau)$, where $\tau \leq t$. This makes sense under the assumption that $s(t)$ can only be observed at time $t$.

**Definition 2 (ReRiSITL Satisﬁability):** For a given $x$, an ReRiSITL formula $\phi$ is said to be satisﬁable if $\forall s \in \mathcal{F}(\mathbb{R}_{\geq 0}, [0,1])$, there exists a nonanticipative strategy $x_{na} : \mathcal{F}(\mathbb{R}_{\geq 0}, [0,1]) \to \mathcal{F}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ s.t. $(x_{na}(s), s, X, 0) \models \phi$.

Later in the article, we will replace risk predicates by deterministic predicates as originally used in STL. For a given constant $c \in \mathbb{R}$, the truth value of such a deterministic predicate $\mu_{det} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{B}$ at time $t$ is obtained as

$$\mu_{det}(x(t), \mu) := \begin{cases} \top & \text{if } h(x(t), \mu) \geq c \\ \bot & \text{otherwise} \end{cases}$$

where we have replaced $X$ in $h$ by its expected value $\mu$.

If now all risk predicates $\mu_{RI} \in M^{RI}$ are replaced by deterministic predicates $\mu_{det}$, then $\phi$ is called a reactive signal interval temporal logic (ReSITL) formula. If uncontrollable propositions $\mu_{sc}$ are excluded, i.e., $M^{sc} = \emptyset$, then $\phi$ is called a risk signal interval temporal logic (RISITL) formula. If all risk predicates are replaced by deterministic predicates and $M^{sc} = \emptyset$, then $\phi$ reduces to an STL formula as in [1].

### B. From MITL to Timed Signal Transducer

We next deﬁne MITL [28] which has the advantage that it can be translated into a timed signal transducer [32]. We later interpret ReRiSITL formulas as MITL formulas and make use of this translation. Instead of predicates and uncontrollable propositions, MITL considers (controllable) propositions $p \in AP$, where $AP$ is a set of atomic propositions. The MITL syntax is hence

$$\varphi := \top \mid p \mid \neg \varphi \mid \varphi' \land \varphi'' \mid \varphi' U \varphi'' \mid \varphi' U_I \varphi''$$

where $\varphi'$ and $\varphi''$ are MITL formulas. Let $d : \mathbb{R}_{\geq 0} \to [0,1]^{|AP|}$ be a Boolean signal corresponding to truth values of $p \in AP$ over time. Define again the projection of $d$ onto $p \in AP$ as $\text{proj}_p(d) : \mathbb{R}_{\geq 0} \to \mathbb{B}$ and let $(d, t) \models \varphi$ be the satisfaction relation. The continuous-time semantics of an MITL formula [32, Sec. 4] are defined as $(d, t) \models p$ if $\text{proj}_p(d)(t) = \top$ while the other operators are as in Definition 1. An MITL formula $\varphi$ is satisﬁable if $\exists d \in \mathcal{F}(\mathbb{R}_{\geq 0}, [0,1]^{|AP|})$, such that $(d, 0) \models \varphi$. Note that the symbols $\varphi$ and $\psi$ are used to distinguish between MITL and ReRiSITL formulas, respectively.

The translation of $\varphi$ into a timed signal transducer is summarized next [32]. Let $c := [c_0, \ldots, c_n]^T \in \mathbb{R}^n_0$ be a vector of $O$ clock variables that obey the continuous dynamics $\dot{c}_o(t) := 1$ with $c_o(0) := 0$ for $o \in \{1, \ldots, O\}$. Discrete dynamics occur at instantaneous times in form of clock resets. Let $r : \mathbb{R}_{\geq 0} \to \mathbb{R}^O_{\geq 0}$ be a reset function, such that $r(c) = c'$ where either $c_o = c_o'$ or $c_o = 0$. With a slight abuse of notation, we use $r(c_o) = c_o$ and $r(c_o) = 0$. Clocks evolve with time when visiting a state of a timed signal transducer, while clocks may be reset during transitions between states. We deﬁne clock constraints as Boolean combinations of conditions of the form $c_o \leq k$ and $c_o \geq k$ for some $k \in \mathbb{Q}_{\geq 0}$. Let $\Phi(c)$ denote the set of all clock constraints over clock variables in $c$.

**Definition 3 (Timed Signal Transducer):** A timed signal transducer is a tuple $TST := (S, S_0, \Delta, \Gamma, c, \ell, \Delta, \gamma, \alpha)$, where $S$ is a ﬁnite set of states, $S_0$ is the initial state with $S_0 \cap S = \emptyset$, and $\Gamma$ are a ﬁnite sets of input and output variables, respectively, $\ell : S \to \Phi(c)$ assigns clock constraints over $c$ to each state, $\Delta$ is a transition relation so that $\delta = (s, g, r, s') \in \Delta$ indicates a transition from $s \in S \cup S_0$ to $s' \in S$ satisfying the guard constraint $g \subseteq \Phi(c)$ and resetting the clocks according to $r : \Delta \to BC(\Delta)$ and $\gamma : \Delta \to BC(\Gamma)$ are input and output labeling functions, where $BC(\Delta)$ and $BC(\Gamma)$ denote the sets of all Boolean combinations over $\Delta$ and $\Gamma$, respectively, and $\Delta \subseteq 2^{S \cup S_0}$ is a generalized Büchi acceptance condition.

A run of a TST over an input signal $d : \mathbb{R}_{\geq 0} \to [0,1]^{|\Gamma|}$. A time step of duration $\tau \in \mathbb{R}_{\geq 0}$ is denoted by $(s, c(t)) \rightarrow (s, c(t) + \tau)$ with $d(t + \tau) = \delta(s) + \gamma(s) + (c(t) + \tau)$ for each $t' \in (0, \tau)$. A discrete step at time $t$ is denoted by $(s, c(t)) \rightarrow (s, c(t) + \delta(s, r(c(t))))$ for some transition $\delta = (s, g, r, s') \in \Delta$, such that $d(t) = \delta(s) + \gamma(s)$, and $d(t) = \gamma(s)$. Each run starts with a discrete step from the initial conﬁguration $(s_0, c(0))$. Formally, a run of a TST over $d$ is a sequence $(s_0, c(0)) \rightarrow (s_1, c(0)) \rightarrow (s_2, c(0)) \rightarrow (s_3, c(0)) \rightarrow \cdots$. Due to the alternation of time and discrete steps, the signals $d(t)$ and $y(t)$ may be a concatenation of sequences consisting of points and open intervals. We associate a function $q : \mathbb{R}_{\geq 0} \rightarrow S \cup \Delta$ with a run as $q(0) := 0 = s_t$ for all $t \in (0, \tau_1), \ldots ; A$ is a generalized Büchi acceptance condition so that a run over $d(t)$ is accepting if, for each $A \in \Phi(c)$, $\inf(q) \cap A \neq \emptyset$ where $\inf(q)$ contains the states in $S$ that are visited, in $q$, for an unbounded time duration and transitions in $\Delta$ that are taken, in $q$, inﬁnitely many times. The language of TST is $L(TST) := \{d \in \mathcal{F}(\mathbb{R}_{\geq 0}, [0,1]^{|\Gamma|}) \mid \text{has an accepting run over } d(t)\}$. The synchronous behavior of two timed signal transducers $TST_1$ and $TST_2$ is deﬁned by their synchronous product $TST_1 \parallel TST_2$. The input–output behavior of $TST_1$ being the input of $TST_2$ is denoted by their input–output composition $TST_1 \triangleright TST_2$, see [32] and [17, Def. 2 and 3] for deﬁnitions.

We can now summarize the procedure of [32]. First, it is shown that every MITL formula $\varphi$ can be rewritten using only the temporal operators $U(0,\infty), U(I,0,\infty)$, $F(0,b)$, and $E(0,b)$ for.
Given a random variable $x$ need to be applied to the basic timed signal transducers of the.

The functions $d, d_1, d_2$ here are used as generic input symbols, while $y$ is a generic output symbol. Fig. 2(e) shows the formula tree for the MITL formula $\varphi := F_{(0,5)} \neg p_1 \lor (p_2 U_{(0,\infty)} p_3 \land F_{(0,15)} p_4)$. To construct the timed signal transducer $TST_{\varphi}$ for $\varphi$ from the formula tree, the synchronous product operation $\sqcap$ and the input–output composition operation $\triangleright$ need to be applied to the basic timed signal transducers of the blocks in the formula tree as indicated in Fig. 2(e).

Third, the formula tree of an MITL formula $\varphi$ is constructed as illustrated in Fig. 2(e). Fourth, input–output composition $\triangleright$ and the synchronous product $\sqcap$ are used to obtain a timed signal transducer $TST_{\varphi} := (S, s_0, A, \Gamma, \epsilon, \Delta, \lambda, \gamma, \Delta_0)$ with $A := AP$ and $\Gamma := \{y\}$; $TST_{\varphi}$ has accepting runs over $d$, i.e., $d \in L(TST_{\varphi})$, with $y(0) = \top$ if and only if $(d, 0) \models \varphi$ [32, Th. 6.7]. Note that $y(0) = \top$ (meaning that $\gamma(\delta_0) = y$, where $\delta_0$ is the initial transition) indicates satisfaction of $\varphi$ at time $t = 0$, while $y(0) = \bot$, i.e., $\gamma(\delta_0) = \neg y$, indicates $(d, 0) \not\models \varphi$.

C. Problem Definition

The first problem is a verification problem to check the satisfiability of an ReRISITL formula $\phi$ according to Definition 2.

**Problem 1:** Given a random variable $X$ and an ReRISITL formula $\phi$ as in (2), check whether or not $\phi$ is satisfiable.

The second problem is a control problem. Let the system

$$\dot{x}(t) = f(x(t)) + g(x(t))u, \quad x(0) := x_0 \tag{5}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are locally Lipschitz continuous and where $u \in \mathbb{R}^m$ is a control law.

In this context, $X$ and $M^{uc}$ may model the environment in which the system in (5) operates, e.g., regions of interest and sensor failures can be modeled by $X$ and $M^{uc}$, respectively. Let now each $\mu_m \in M^{Ri}$ with $m \in \{1, \ldots, |M^{Ri}|\}$ be associated with predicate functions $h_m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and risk parameters $R_m(\cdot), \beta_m$, and $\gamma_m$. For $\mu^{uc} \in M^{uc}$, let the truth value of $\mu^{uc}$ at time $t \in \mathbb{R}_{\geq 0}$ be captured by $s \in \mathcal{F}(\mathbb{R}_{\geq 0}, \mathbb{B}^{|M^{uc}|})$, i.e., we observe $\text{proj}_{\mu^{uc}}(s)(t)$. Since $s$ is not known beforehand, we assume to observe $s(t)$ at time $t$.

**Problem 2:** Given a random variable $X$ and a satisfiable ReRISITL formula $\phi$ as in (2), find a nonanticipative strategy $u(x(t), s, t)$ s.t. $(x, s, X(0)) \models \phi$, where $x$ is the solution to (5) under $u(x(t), s, t)$ and where $x(t)$ is observed at time $t$.

The next assumption is not explicitly used and needed for our proposed solutions to Problems 1 and 2. We will, however, refer to this assumption in some places to emphasize that computational advantages can be obtained under it.

**Assumption I:** The functions $h_m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are linear in its first argument.

III. SATISFIABILITY OF ReRISITL SPECIFICATIONS

In this section, we present a solution to Problem 1. In Sections III-A and III-B, we construct a timed signal transducer $TST_{\phi}$ that characterizes all signals $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and...
\( s : \mathbb{R}_{\geq 0} \to \mathbb{B}^{[M^r]} \), such that \( (x, s, X, 0) \models \phi \). In Section III-C, we consider if, for all \( s \in F(\mathbb{R}_{\geq 0}, \mathbb{B}^{[M^r]}) \), there exists a non-anticipative strategy \( x_{na} : F(\mathbb{R}_{\geq 0}, \mathbb{B}^{[M^r]}) \to F(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \), such that \( (x_{na}(s), s, X, 0) \models \phi \), solving Problem 1.

A. From ReRiSITL to Timed Signal Transducer

The first goal is to abstract the ReRiSITL formula \( \phi \) into an MITL formula \( \varphi \) via a transformation \( T \hat{r}(\cdot) \). Therefore, let us use the notation \( \varphi(M) \) to make explicit that the ReRiSITL formula \( \phi \) depends on the set of predicates and propositions \( M \). The transformation \( T \hat{r}(\cdot) \) essentially replaces predicates and uncontrollable propositions \( M \) in \( \phi(M) \) by a set of propositions \( A.P. \). For \( i \in \{1, \ldots, |M|\} \), associate with each \( \mu_i \in M \) a proposition \( p_i \), and let \( A.P. := \{p_1, \ldots, p_M\} \). Then let \( \varphi := T \hat{r}(\varphi(M)) = \phi(A.P.) \), e.g., \( \varphi(A.P.) = F_I(p_1 \land p_2) \). Let the inverse \( T^{-1}(\varphi) = T^{-1}(T \hat{r}(\varphi(M))) = \phi(M) \) be obtained by replacing each \( p_i \in A.P. \) in \( \varphi(M) \) with the corresponding \( \mu_i \in M \).

Let now \( TST_{\varphi} := (S, s_0, \Lambda, c, t, \Delta, \lambda, \gamma, A) \) be constructed for the MITL formula \( \varphi \) according to Section II-B with \( \Lambda := A.P. \). Since we aim at satisfying the STL formula \( \phi \), we modify \( TST_{\varphi} \) by the following operations to account for the error induced by the abstraction from \( \phi \) to \( \varphi \) via \( T \hat{r} \).

1. Remove each state \( s \in S \) for which there exists no \( x \in \mathbb{R}^n \) and no \( s \in B^{[M^r]} \) so that \( (x, s, X) \models T r^{-1}(\lambda(s)) \). Remove the corresponding \( s \) from \( A. \) Further remove the corresponding ingoing (\( (s', g, r, s) \in \Delta \) for some \( s' \in S \)) and outgoing (\( (s, g, r, s') \in \Delta \) for some \( s' \in S \)) transitions.

2. Remove each transition \( \delta := (s, g, r, s') \in \Delta \) for which there exists no \( x \in \mathbb{R}^n \) and no \( s \in B^{[M^r]} \) so that \( (x, s, X) \models T r^{-1}(\lambda(\delta)) \). Remove the corresponding \( \delta \) from \( A. \)

The modified \( TST_{\varphi} \) is denoted by \( TST_{\phi} := (S^\phi, s_0, \Lambda, c, t, \Delta^\phi, \lambda, \gamma, A^\phi) \), for which naturally \( S^\phi \subseteq S, \Delta^\phi \subseteq \Delta, \) and \( A^\phi \subseteq A \). Note that it is essential to be able to check if there exists \( x \in \mathbb{R}^n \) and \( s \in B^{[M^r]} \) such that \( (x, s, X) \models T r^{-1}(\lambda(s)) \). Therefore, we will obtain computationally more efficient mixed integer linear programs if Assumption 1 holds.

B. Satisfiability of RiSITL Specifications

To characterize all signals \( x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \) and \( s : \mathbb{R}_{\geq 0} \to B^{[M^r]} \) so that \( (x, s, X, 0) \models \phi \), we translate \( TST_{\phi} \) of the previous subsection, which is in essence a timed automaton when removing the output labels, to a region automaton \( RA(TST_{\phi}) \) [29].

The intuition of a plan \( d : \mathbb{R}_{\geq 0} \to BC(\Lambda) \) is as follows: A signal \( d : \mathbb{R}_{\geq 0} \to B^{[M^r]} \) that satisfies the plan \( d \) also satisfies the MITL specification \( \varphi \) at time \( t = 0 \), i.e., \( d(t) = d_p(t) \) for all \( t \geq 0 \) implies that \( (d, 0) \models \varphi \).
Lemma 1: Given a signal $d : \mathbb{R}_{\geq 0} \to \mathbb{B}^{\left|AP\right|}$, there is an accepting run of $TST_{\phi}$ over $d(t)$ and $(d, 0) \models \phi$ if only if there exists a plan $d_p(t)$ so that $d(t) \models d_p(t)$ for all $t \in \mathbb{R}_{\geq 0}$.

Proof: $\Rightarrow$: Departing from $TST_{\phi}$, the infinite state transition system $(S^0 \times \mathbb{R}^0_{\geq 0})$ has, by construction, the same reachable set as $TST_{\phi}$, i.e., the same reachable configurations $(a_0, c(0)), (a_0, r(c(0))), (a_1, c(0)) + \tau_1, \ldots$. Since $\sim$ is a bisimulation relation, reachability properties of $TST_{\phi}$ can then equivalently be analyzed by considering the finite state transition system $RA(TST_{\phi})$ [29, Lemma 4.13]. If there hence exists an accepting run of $TST_{\phi}$ over $d(t)$ and $(d, 0) \models \phi$, i.e., $(d_0, y) = y$, the plan $d_p(t)$ can be constructed as described above by obtaining $q$ and $\tau$ directly from the accepting run of $TST_{\phi}$ over $d(t)$. It will, by construction, hold that $d(t) \models d_p(t)$ for all $t \in \mathbb{R}_{\geq 0}$.

$:=$: If there exists a plan $d_p(t)$ so that $d(t) \models d_p(t)$ for all $t \in \mathbb{R}_{\geq 0}$, then it follows that $TST_{\phi}$ has an accepting run over $d(t)$.

By construction of $d_p(t)$, where $q$ and $\tau$ have been obtained based on $RA(TST_{\phi})$ (as described for the synthesis of $d_p(t)$) and by the bisimulation relation $\sim$. Removing states and transitions from $TST_{\phi}$ according to operations $[O1]$ and $[O2]$ resulting in $TST_{\phi}$ only removes behavior from $TST_{\phi}$ (not adding additional behavior), i.e., $L(TST_{\phi}) \subseteq L(TST_{\phi})$, so that, by [32, Th. 6.7], an accepting run of $TST_{\phi}$ over $d(t)$ inducing $y(0) = \tau$ results in $(d, 0) \models \phi$.

Note that there may exist an accepting run of $TST_{\phi}$ over $d(t)$ so that $(d, 0) \models \phi$, while there exists no accepting run of $TST_{\phi}$ over $d(t)$ due to operations $[O1]$ and $[O2]$. We can now associate $d_p : \mathbb{R}_{\geq 0} \to BC(M)$ with $d_p(t)$ as

\[ d_p(t) := Tr^{-1}(d_p(t)) \]

and, based on $\phi$, state under which conditions $d_p(t)$ exists.

Theorem 1: There exists a plan $d_p(t)$ (and hence a plan $d_p(t)$) if and only if there exists a $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $s : \mathbb{R}_{\geq 0} \to \mathbb{B}^{\left|M^\text{uc}\right|}$ so that $(x, s, 0, 0) \models \phi$.

Proof: $\Rightarrow$: The existence of a plan $d_p(t)$ implies, by Lemma 1, that a signal $d : \mathbb{R}_{\geq 0} \to \mathbb{B}^{\left|AP\right|}$ with $d(t)$ an $d_p(t)$ for all $t \in \mathbb{R}_{\geq 0}$ is such that $(d, 0) \models \phi$. Operations $[O1]$ and $[O2]$ remove all states and transitions $\delta$ from $TST_{\phi}$ that are infeasible, i.e., for which there exists no $x \in \mathbb{R}^n$ and no $s \in \mathbb{B}^{\left|M^\text{uc}\right|}$ such that $(x, s, X) \models Tr^{-1}(\lambda(s))$ and $(x, s, X) \models Tr^{-1}(\lambda(\delta))$, respectively. Recall that the only difference between the semantics of $\phi$ and $\varphi$ is the difference in $\mu_1$ and $\mu_2$, respectively. It follows that, based on the run of $TST_{\phi}$ over $d(t)$, we can construct a signal $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $s : \mathbb{R}_{\geq 0} \to \mathbb{B}^{\left|M^\text{uc}\right|}$ with $(x(t), s(t), X) \models d_p(t)$ for all $t \in \mathbb{R}_{\geq 0}$ implying that $(x, s, 0, 0) \models \phi$.

$:=$: Based on $(x(t)$ and $s(t)$, define the signal $d(t) := (h_1^T(x(t)) \ldots h_{\left|M^\text{uc}\right|}^T(x(t))) s(t)^T$, where $h_m(x) := \top$ if $\gamma_m h_m(x, \mu) \leq 0$ and $h_1^T(x) := \bot$ otherwise and that is such that $(d, 0) \models \phi$. Note that $h_m(x, \mu)$ is the predicate function associated with $\mu_m$. It follows that $d$ induces an accepting run of $TST_{\phi}$ over $d(t)$ since the traversed states and transitions during this run have not been removed by operations $[O1]$ and $[O2]$. By Lemma 1, it follows that there hence exists a plan $d_p(t)$.

The next two results are straightforward consequences.

Corollary 1: If $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $s : \mathbb{R}_{\geq 0} \to \mathbb{B}^{\left|M^\text{uc}\right|}$ are so that $(x(t), s(t), X) \models d_p(t)$ for all $t \in \mathbb{R}_{\geq 0}$, then it follows that $(x, s, 0, 0) \models \phi$.

Algorithm 1: Algorithm to Check if $\phi$ is Satisfiable.

1: Obtain the MITL formula $\phi := Tr(\phi)$.
2: Obtain $TST_{\phi}$ according to Section II-B and where uncontrollable propositions $p_i \in AP$, i.e., $p_i$ with $Tr^{-1}(p_i) \in M \cap M^\text{uc}$, are modeled as in Fig. 3.
3: Perform $[O1]$ and $[O2]$ to obtain $TST_{\phi}$.
4: Modify $TST_{\phi}$ to avoid Zeno behavior.
5: Translate $TST_{\phi}$ into $RA(TST_{\phi})$.
6: Translate $RA(TST_{\phi})$ into $\overline{RA}(TST_{\phi})$.
7: Run Algorithm 2 to obtain $W$.
8: Check if the conditions in Theorem 2 are satisfied.

Corollary 2: If $M^\text{uc} = \emptyset$, i.e., $\phi$ is an RiSITL formula, then it holds that there exists a plan $d_p(t)$ (and hence a plan $d_p(t)$) if and only if $\phi$ is satisfiable.

C. Satisfiability of ReRiSITL Specifications

The previous results can only be used to check satisfiability of RiSITL. For ReRiSITL specifications $\phi$, this requires to check all $s \in \mathcal{F}(\mathbb{R}_{\geq 0}, \mathbb{B}^{\left|M^\text{uc}\right|})$ as in Definition 2. Let us define

\[ s^T := [\bot \ldots \bot]^T \in \mathbb{B}^{\left|M^\text{uc}\right|} \]

and additionally impose the following assumption that all signals $s \in \mathcal{F}(\mathbb{R}_{\geq 0}, \mathbb{B}^{\left|M^\text{uc}\right|})$ have to satisfy.

Assumption 2: Assume that $s(t) = s^T$ for all times except on a set of measure zero, i.e., $s(t) \neq s^T$ only for a countable set of times $t$. There exists a known lower bound $\zeta > 0$ between events $s(t) \neq s^T$, i.e., for $s(t') = s(t') \neq s^T$ with $t' \neq t''$, it holds that $|t'' - t'| \geq \zeta$.

Assumption 2 excludes signals $s(t)$ exhibiting Zeno behavior, i.e., infinite changes of $s(t)$ in finite time, and is realistic in the sense that it allows to model instantaneous error signals such as considered for communication dropouts or sensor failures. Assumption 2 is in particular necessary for a game-based approach, see [33]. Furthermore, Assumption 2 is necessary for the replanning procedure in Section V-B.

In Algorithm 1, presented below and explained in the remainder, we summarize the steps to check if $\phi$ is satisfiable. Line 1 in Algorithm 1 has already been explained, while line 2 is related to Assumption 2. In particular, to model uncontrollable propositions $p_i : AP$ according to Assumption 2, we consider the timed signal transducer in Fig. 3. When constructing $TST_{\phi}$, we hence model each $p \in AP$ with $\mu_{\text{uc}} = Tr^{-1}(p) \in M^\text{uc}$ as in Fig. 3. Line 3 in Algorithm 1 then performs $[O1]$ and $[O2]$ to obtain $TST_{\phi}$.

Within the presented game-based approach, it needs to be ensured that no player (here the two players are the controllable and uncontrollable signals $x$ and $s$) wins by inducing Zeno behavior (see [33] for more intuition). A generic way of avoiding
Zeno behavior is to add a clock $c$ to TST$_\phi$ and add, to each transition, the constraint $c \geq \epsilon$ for a small constant $\epsilon \in \mathbb{Q}_{>0}$ and the reset function $r(c) := 0$. This modification will affect the completeness, but not the soundness of the proposed approach. There are minimally invasive algorithms on how to avoid Zeno behavior, for instance as in [51]. This modification of TST$_\phi$ is stated in line 4 in Algorithm 1.

Recall from Section II-B that an accepting run in TST$_\phi$ needs to satisfy the generalized Büchi acceptance condition which implies having infinite length, i.e., the run is not allowed to stop existing. The latter is necessary since we require to be able to extend each finite run in TST$_\phi$ to an infinite run. Specifically, note that within a state $s \in S^\phi$ in TST$_\phi$ it may happen that, for some $s \in \mathbb{B}^{\mathcal{M}_i}$, there exists no $x \in \mathbb{R}^n$, such that a transition can be taken, i.e., there exists no $\delta' := (s, g', r', s') \in \Delta^\phi$ such that $(x, s, X) = \lambda(\delta')$. This means that there is no continuation of a finite run entering the state $s$ so that the run is not accepting. For instance, in Fig. 2(b) in the bottom right state there exists no transition for $\text{proj}_{\delta}(d) = \top$. To account for this, we first modify the infinite state transition system $(S^\phi \times \mathbb{R}^n_0, \Rightarrow)$ to $(S^\phi \times \mathbb{R}^n_0, \Rightarrow_c)$ by separating time and discrete transitions.

**Definition 6 (Equivalent Transition System of TST$_\phi$):** Let $(S^\phi \times \mathbb{R}^n_0, \Rightarrow_c)$ be a transition system, where $(s, c) \overset{\delta_c}{\Rightarrow} (s', c')$ with $\delta_t \in \{\delta, t\}$ if there is either a discrete or a time transition as follows:

1. There is a discrete transition $(s, c) \overset{\delta_c}{\Rightarrow} (s', c')$ if there exists $\delta := (s, g, r, s') \in \Delta^\phi$ so that $c' = r(c)$ and $g = g$;
2. There is a time transition $(s, c) \overset{\delta_c}{\Rightarrow} (s', c')$ if, for all $\tau \in (0, t)$, $c + \tau \overset{\delta}{\Rightarrow} (s)$.

We emphasize that $(S^\phi \times \mathbb{R}^n_0, \Rightarrow_c)$, $(S^\phi \times \mathbb{R}^n_0, \Rightarrow)$, and hence TST$_\phi$ have the same reachability properties. Let now $RA(C(TST_\phi)) := (Q, q_0, \Delta_R, A_R)$ denote the region automaton, similar to Definition 5, but now obtained from $(S^\phi \times \mathbb{R}^n_0, \Rightarrow_c)$ instead of $(S^\phi \times \mathbb{R}^n_0, \Rightarrow)$. The translation to $RA(C(TST_\phi))$ corresponds to line 5 in Algorithm 1.

**Definition 7 (Region Automaton of TST$_\phi$):** The region automaton $RA(C(TST_\phi)) := (Q, q_0, \Delta_R, A_R)$ is defined as follows.

1. The states are $q := (s, \alpha)$, where $s \in S^\phi$ and $\alpha \in A$ where $A$ is the set of all clock regions so that $Q := S^\phi \times A$.
2. The initial states are $q_0 := (s_0, \alpha_0)$ in $Q$, where $\alpha_0$ is the clock region corresponding to $c(0)$.
3. For $q := (s, \alpha)$ and $q' := (s', \alpha')$, there is a transition $(q, \delta_t, q') \in \Delta_R$, where $\delta_t \in \{\delta, t\}$ if there is
   a) either a discrete transition $(s, c) \overset{\delta_c}{\Rightarrow} (s', c')$ for $c \in \alpha$ and $c' \in \alpha'$;
   b) or a time transition $(s, c) \overset{\delta_c}{\Rightarrow} (s', c')$ for $c \in \alpha$ and $c' \in \alpha'$, where $\alpha'$ is the immediate time successor of $\alpha$.
4. $q = (s, \alpha) \in A_R(i)$ if $s \in A^\phi(i)$.

**Remark 3:** Defining $RA(C(TST_\phi))$ based on $(S^\phi \times \mathbb{R}^n_0, \Rightarrow_c)$ by separating discrete and time transitions, and unrolling the time domain as in Definition 7, results in more states compared to $RA(TST_\phi)$ based on $(S^\phi \times \mathbb{R}^n_0, \Rightarrow)$. This, however, now becomes necessary since uncontrollable signals $s$ may cause undesirable behavior at all times.

Translate now $RA(C(TST_\phi))$, which is a finite automaton with generalized Büchi acceptance condition, into an equivalent finite automaton $\overline{RA}(C(TST_\phi)) := (\overline{Q}, \overline{q}_0, \overline{\Delta}_R, \overline{A}_R)$ with a Büchi acceptance condition instead, as follows.

1. $\overline{Q} := Q \times \{1, \ldots, |A_R|\}$.
2. $\overline{q}_0 := (q_0, 1)$.
3. $\overline{\Delta}_R := \{(i, j, \delta_t, (q, j'))|(i, \delta_t, q') \in \Delta_R \text{ and if } q \in A_R(i), \text{ then } j = ((i + 1) \mod |A_R| + 1) \text{ else } j = i\}$, where $A_R(i)$ denotes the $i$th element of $A_R$.
4. $\overline{A}_R := (A_R(1), 1)$.

In particular, the difference is that $\overline{A}_R$ consists of several sets $A_R(i)$ of states, while $A_R(i)$ is a single set of states. By construction, the accepting behavior of $RA(C(TST_\phi))$ and $\overline{RA}(C(TST_\phi))$ are the same. This translation corresponds to line 6 in Algorithm 1 and is performed to obtain a simpler acceptance condition that can be expressed as a fixed point expression as we will see below. In fact, a winning condition (for a game played between $s$ and $x$) is that always eventually $A_R$ can be visited by each finite run of $\overline{RA}(C(TST_\phi))$.

**Remark 4:** The translation to $\overline{RA}(C(TST_\phi))$ may induce $|Q| \cdot |A_R|$ states. One can avoid such a state explosion by neglecting the acceptance condition $A$ for all timed signal transducers in Fig. 2 except for the until operator in Fig. 2(a), where a Büchi acceptance condition is needed.

In the remainder, we are inspired by the work in [33]. We first introduce the main operator, the controllable predecessor $\pi : 2^\overline{Q} \rightarrow 2^\overline{Q}$. For a certain set $W \subseteq \overline{Q}$, define

$$\pi(W) := \{\overline{q} \in \overline{Q} | \exists s \in \mathbb{B}^{\mathcal{M}_i}, \exists (\overline{q}, \delta, \overline{q}') \in \overline{\Delta}_R \text{ s.t.}$$

1. $\overline{q}' \in W$, 2. $\exists x \in \mathbb{R}^n$ s.t. $(x, s, X) = Tr^{-1}(\lambda(\delta))\}$.

The intuition is that states in $\pi(W)$ will always allow to enforce a transition into $W$ by a suitable $x$ in one step, no matter of the value of $s$. We next present Algorithm 2 to obtain the set $W$ from which we can force to always eventually be within $\overline{A}_R$. Algorithm 2, called in line 7 in Algorithm 1, differs from the algorithm presented in [33] by the definition of the controllable predecessor $\pi : 2^\overline{Q} \rightarrow 2^\overline{Q}$.

The algorithm starts with $W_0 := \overline{Q}$ (line 1). For this $W_0$, the inner loop (lines 3–5) calculates all states $H_j$ from which states in $\overline{A}_R \cap \pi(W_0)$ can be reached, i.e., states in $\overline{A}_R$ that can be reached and are no deadlock states. For $W_1 := H_j$ (line 6), this inner loop is repeated until eventually obtaining the set of states $W$ that can always eventually be reached.
The set $W$ tells us if we can let time pass or if a transition according to $\pi(W)$ has to be taken in a particular state. For $TST_\phi$ restricted to $W$ this means that, at no time, an uncontrollable proposition $\phi$ can force the system into a state from where the Büchi acceptance condition cannot be satisfied. The operator $\pi(W)$ then determines which $x$ can be selected in case of a particular $s$. Note in particular, as similarly analyzed in [33], that $W_i$ in Algorithm 2 is monotonically decreasing, such that a fixed point, i.e., $W_{i+1} = W_i$, is eventually reached, such that Algorithm 2 terminates in a finite number of steps.

**Theorem 2:** If $s$ is according to Assumption 2, then it holds that the ReRiSITL formula $\phi$ is satisfiable if $\pi_0 \in W$ and if there exists $(q_0, \delta_0, \pi') \in \Delta_R$ with $\gamma(\delta_0) = y$.

**Proof:** First note that due to the use of the timed signal transducer as in Fig. 3, we account for the form of $s$ as in Assumption 2. Recall also from Theorem 1 that operations $[O1]$ and $[O2]$ restrict the behavior of $TST_\phi$ to the signals $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $s : \mathbb{R}_{\geq 0} \to \mathbb{B}^{[M_{ri}]}$ with $(x, s, X, 0) = \phi$. Note that $RA_C(TST_\phi)$ has, by construction, the same reachable set as $TST_\phi$. Recall also that $RA_C(TST_\phi)$ and $\overline{RA_C(TST_\phi)}$ are equivalent so that reachability properties of $TST_\phi$ can equivalently be verified on $\overline{RA_C(TST_\phi)}$. We now need to prove that, for each $s \in F(\mathbb{R}_{\geq 0}, \mathbb{B}^{[M_{ri}]}$) that satisfies Assumption 2, there is an accepting run in $\overline{RA_C(TST_\phi)}$ restricted to the states in $W$ that satisfies the Büchi acceptance condition. By Algorithm 2, which is guaranteed to terminate in a finite number of steps, it is ensured that no state in $W$ is a deadlock and can be continued to another state in $W$. Specifically, it is guaranteed that for each state in $W$ an infinite continuation can be found that satisfies the Büchi acceptance condition, no matter how $s(t)$ behaves. Note also that Zeno winning conditions have been excluded by modifying $TST_\phi$ to not permit Zeno behavior. Since $\pi_0 \in W$ and since there exists $(q_0, \delta_0, \pi') \in \Delta_R$ with $\gamma(\delta_0) = T$, it follows that $\phi$ is satisfiable in the sense of Definition 2.

Note also that Theorem 2 is sufficient. Necessity does not hold due to the modification of $TST_\phi$ to avoid Zeno behavior, potentially introducing conservatism.

Finally, we remark that Sections III-A and III-B use graph search techniques, while Section III-C follows a game-based approach. One could argue that only the game-based approach solving Problem 1 is of interest. We have, however, chosen this particular exposition of our results since we will combine graph search techniques with a game-based approach to address Problem 2 in the following Sections IV and V.

**IV. FROM ReRiSITL TO ReSITL BY DETERMINIZING RISK PREDICATES**

Fig. 4 can be used as a guide in the remainder of the article as it shows an overview of the reactive planning and control strategy that will be presented in Sections IV and V. Starting in the top right box of Fig. 4, this section introduces the idea to determinize risk predicates in $M^R_n$ and replace them with deterministic predicates, hence converting the ReRiSITL formula $\phi$ into an ReSITL formula $\theta$ that we then deal with in Section V. We provide conditions under which a certain soundness property holds which ensures that satisfaction of $\theta$ implies satisfaction of $\phi$. Sections IV-A and IV-B assume that $\phi$ is in positive normal form. In the end of Section IV-B, we discuss how we can deal with situations where this is not the case.

**A. Risk Constrained Sets**

In the following two sections, we will define risk-tightened deterministic predicates $\mu^d_n$ that will replace the risk predicates $\mu^m_n$ and allow for the use of existing control methods. Note that $R((-h_m(x, X)) \text{ depends on } x \in \mathbb{R}^n \text{ (we drop the dependence of } x(t) \text{ on } t \text{ in this section for convenience). For given } \beta_m \in (0, 1) \text{ and } \gamma_m \in \mathbb{R}, \text{ define the sets}$$X_{m}^{EV}(\gamma_m) := \{x \in \mathbb{B} \| EV[\neg h_m(x, X)] \leq \gamma_m\}$$X_{m}^{VaR}(\beta_m, \gamma_m) := \{x \in \mathbb{B} \| VaR_{\beta_m}(-h_m(x, X)) \leq \gamma_m\}$$X_{m}^{CVaR}(\beta_m, \gamma_m) := \{x \in \mathbb{B} \| CVaR_{\beta_m}(-h_m(x, X)) \leq \gamma_m\}.$

Note the set $\mathbb{B} \subseteq \mathbb{R}^n$ that is supposed to be an arbitrarily big compact and convex set as will further be explained in Section V-C. The set $\mathbb{B}$ can be seen as the workspace that (5) will be forced to remain within. The sets $X_{m}^{EV}(\gamma_m), X_{m}^{VaR}(\beta_m, \gamma_m)$, and $X_{m}^{CVaR}(\beta_m, \gamma_m)$ define all $x$ for which the EV, VaR, and CVaR of $-h_m(x, X)$ is less or equal than $\gamma_m$, respectively. If these sets are empty, the underlying predicate is not satisfiable. For $\epsilon_m \in \mathbb{R}$, which is a design parameter as opposed to $\beta_m$ and $\gamma_m$,
define \[ X_m(c_m) := \{ x \in B | h_m(x, \bar{\mu}) \geq c_m \} \]
where the mean \( \bar{\mu} \) has been used instead of \( X \) to evaluate the predicate function \( h_m \). Note that \( X_m(c_m) \) is a compact and convex set if Assumption 1 holds. If \( X_m(c_m) \supseteq X_m(c_m) \), then \( x \in X_m(c_m) \) implies \( x \in X_m(c_m) \) (similarly for \( X_m(\beta_m, \gamma_m) \) and \( X_m(\beta_m, \gamma_m) \)). This implies that predicates within an ReRiSITL formula can be determined by using \( h_m(x, \bar{\mu}) \geq c_m \) [recall (3)] instead of \( R(-h_m(x, X)) \leq \gamma_m \) by conserving an important soundness property (Section IV-B). For given \( c_m \), checking these set inclusions may be nonconvex. As shown in [45, Lemma 1], when \( h_m(x, X) \) is linear in \( x \), this can be checked efficiently since the distribution of \( h_m(x, X) \) is only shifted.

Lemma 2 [45, Lemma 1]: Assume that \( h_m(x, X) = \nu^T x + h^T(x) \) for \( \nu \in \mathbb{R}^n \) and for \( h^T : \mathbb{R}^n \to \mathbb{R} \), then
\[ X_m(c_m) \supseteq X_m(c_m) \text{ iff } EV[-h_m(x, X)] \leq \gamma_m \]
\[ X_m(\beta_m, \gamma_m) \supseteq X_m(c_m) \text{ iff } \text{VaR}_{\beta_m}(-h_m(x, X)) \leq \gamma_m \]
\[ X_m(\beta_m, \gamma_m) \supseteq X_m(c_m) \text{ iff } \text{CVaR}_{\beta_m}(-h_m(x, X)) \leq \gamma_m \]
where \( x^* := \text{argmin}_{x \in X_m(c_m)} \nu^T x \) (a convex problem).

We remark that in particular \( \text{VaR}_{\beta_m}(-h_m(x, X)) \) and \( \text{CVaR}_{\beta_m}(-h_m(x, X)) \) can be efficiently computed [41, Th. 1] and that \( \text{VaR}_{\beta}(-h_m(x, X)) \) is obtained as a byproduct of the calculation of \( \text{CVaR}_{\beta}(-h_m(x, X)) \). If \( h_m(x, X) \) is nonlinear, we argue that, for some function classes, numerical methods can be used to check these set inclusions, e.g., when \( h_m(x, X) \) is quadratic in \( x \).

B. Converting ReRiSITL Into ReSITL Specifications

Considering the ReRiSITL formula \( \phi \) that consists of the risk predicates \( \mu_m \in M^R \) with \( m \in \{1, \ldots, |M|^R| \} \), we transform the ReRiSITL formula \( \phi \) into an ReSITL formula \( \theta \). In particular, \( \theta \) is obtained by replacing risk predicates \( \mu_m \in M^R \) in \( \phi \) by a deterministic predicate \( \mu_m^d \) according to (3). More formally and by denoting \( \phi(M^R, M^w) \) instead of \( \phi \) to highlight the dependence on risk predicates \( M^R \) and uncontrollable propositions \( M^w \), let \( \theta := \phi(M^d, M^w) \), where
\[ M^d := \{ \mu_1^{d}, \ldots, \mu_{|M|^R}^{d} \}. \]

Let now \( \tilde{M} := M^d \cup M^w \) and let us associate the semantics \( (x, s, \bar{\mu}, t) \models \theta \) with an ReSITL formula \( \theta \). The next assumption is sufficient to ensure soundness in the sense that \( (x, s, \bar{\mu}, t) \models \theta \) implies \( (x, s, X, t) \models \phi \).

Assumption 3: For each \( m \in \{1, \ldots, |M|^R| \} \), \( X_m(c_m) \supseteq X_m(c_m) \), \( X_m(\beta_m, \gamma_m) \supseteq X_m(c_m) \), or \( X_m(\beta_m, \gamma_m) \supseteq X_m(c_m) \) (depending on the type of predicate).

Example 2: By setting \( \epsilon := 0.35 \) for the VaR predicates and \( c := 0.9 \) for the CVaR predicates in Example 1, Assumption 3 is satisfied. The red circles in Fig. 1 indicate the obtained deterministic predicates, based on the predicate functions
\[ h_R^d(x, \bar{\mu}) := \epsilon - \|x - \bar{\mu}_R^d\|^2 - 0.35 \]
\[ h_{R2}^d(x, \bar{\mu}) := \epsilon - \|x - \bar{\mu}_{R2}^d\|^2 - 0.35 \]
\[ h_{O1}^d(x, \bar{\mu}) := \epsilon - \|x - \bar{\mu}_{O1}^d\|^2 - 0.35 \]
\[ h_{O2}^d(x, \bar{\mu}) := \epsilon - \|x - \bar{\mu}_{O2}^d\|^2 - 0.35 \]

Passing in between the obstacles O1 and O2 is not possibly due to the uncertainty in \( X \) and the risk predicates. Increasing \( c_m \) shrinks the set \( X_m(c_m) \) so that Assumption 3 (verifiable by Lemma 2) poses a lower bound on \( c_m \).

Theorem 2: Let Assumption 3 hold and \( \phi \) be an ReSITL formula in positive normal form. If \( x : \mathbb{R}_0 \to \mathbb{B} \) and \( s : \mathbb{R}_0 \to \mathbb{B} \), are, such that \( (x, s, \bar{\mu}, t) \models \theta \), then it follows that \( (x, s, X, t) \models \phi \).

Proof: Due to Assumption 3, \( x \in X_m(c_m) \) implies \( x \in X_m(c_m) \), \( x \in X_m(\beta_m, \gamma_m) \), or \( x \in X_m(\beta_m, \gamma_m) \) depending on the type of the predicate \( m \). It is now straightforward to recursively show on the ReSITL semantiques in Definition 1 that \( (x, s, \bar{\mu}, t) \models \theta \) implies \( (x, s, X, t) \models \phi \) when \( x(t) \in B \), which holds by assumption. This follows since the semantics of ReRiSITL and ReSITL only differ on the predicate level and since negations are excluded since \( \phi \) is in positive normal form.

An important task is to pick the set of \( c_m \). In general, we may induce conservatism since the level sets of \( X_m(c_m) \) may not be aligned with the level sets of \( X_m(c_m) \), \( X_m(\beta_m, \gamma_m) \), or \( X_m(\beta_m, \gamma_m) \). When linearity of \( h_m(x, X) \) in \( x \) holds as in Lemma 2, conservatism can be avoided [45, Lemma 2].

If now, however, \( \phi \) is not in positive normal form, there are two ways how to handle this case. The first way is to find \( c_m \) for each \( m \in \{1, \ldots, |M|^R| \} \) according to [45, Lemma 2], i.e., the set inclusion in Assumption 3 is replaced by an equality. More generally, a more elegant way is to bring \( \phi \) into positive normal form, as for instance shown in [11, Proposition 2]. This would lead to a formula \( \phi \) potentially having negations in front of some or all of the predicates, i.e., \( \gamma_{Ri}^m \). For those predicates, we redefine the sets \( X_m(c_m) \), \( X_m(\beta_m, \gamma_m) \), and \( X_m(\beta_m, \gamma_m) \) as

\[ X_m(c_m) := \{ x \in B | EV[-h_m(x, X)] \leq \gamma_m \} \]
\[ X_m(\beta_m, \gamma_m) := \{ x \in B | VaR_{\beta_m}(-h_m(x, X)) \leq \gamma_m \} \]
\[ X_m(\beta_m, \gamma_m) := \{ x \in B | CVaR_{\beta_m}(-h_m(x, X)) \leq \gamma_m \} \]

Note that only the sign of the inequality has changed compared to the definition in Section IV-A. For \( c_m \in \mathbb{R} \), we then also redefine \( X_m(c_m) \) as
\[ X_m(c_m) := \{ x \in B | h_m(x, X, \bar{\mu}) \leq c_m \} \]
We would now again like to establish the set inclusions as in Assumption 3 by a suitable choice of \( c_m \) with these modified definitions. Note that these inclusions can then be similarly checked as in Lemma 2 (just reversing inequalities again).

V. REACTIVE PLANNING UNDER ReSITL SPECIFICATIONS

Following Section IV, we can obtain an ReSITL formula \( \theta \) from the ReRiSITL formula \( \phi \). Motivated by the soundness result in Theorem 3, we now propose a reactive planning and control method that leads to a satisfaction of the ReRiSITL formula \( \theta \) that consequently leads to the satisfaction of the ReRiSITL formula \( \phi \) (see also the top right box in Fig. 4).

In Section V-A, we abstract the control system in (5) into a timed signal transducer TST \( \tau \) (top left box in Fig. 4).
Algorithm 3: Reactive Planning for ReSITL Formula $\theta$.

1: Obtain the MITL formula $\phi := Tr(\theta)$.  
2: Obtain $\hat{TST}_m$ according to Section II-B and where uncontrollable propositions $p_i \in AP$, i.e., $p_i$ with $Tr^{-1}(p_i) \in M \cap M^{uc}$, are modeled as in Fig. 3.  
3: Perform [O1] and [O2] to obtain $TST_\theta$.  
4: Obtain $TST_c$ according to Section V-A.  
5: Perform [O3], [O4], and [O5] to obtain $TST_m^{\theta}$.  
6: Modify $TST_m^{\theta}$ to avoid Zeno behavior.  
7: Translate $TST_m^{\theta}$ into $RA\hat{C}(TST_m^{\theta})$.  
8: Translate $RA\hat{C}(TST_m^{\theta})$ into $\overline{RA\hat{C}}(TST_m^{\theta})$.  
9: Run Algorithm 2 with the modified function $\pi : 2^Q \rightarrow 2^Q$ and $\overline{RA\hat{C}}(TST_m^{\theta})$ as the inputs to obtain $W$.  
10: Calculate the initial plan $d_\mu(t)$ based on $\overline{RA\hat{C}}(TST_m^{\theta})$ and obtain the associated control law $u(x,t)$ (only possible if the conditions in Theorem 2 are satisfied).  
11: while $s(t) = s^-$ do  
12: if $s(t) \neq s^-$ then  
13: Recalculate $d_\mu(t)$ and $u(x,t)$  
14: Apply $u(x,t)$ to (5)  

abstraction is based on the assumption of existing logic-based feedback control laws from Section V-C. We then modify $TST_\theta$ into $TST_m^{\theta}$ (bottom box in Fig. 4), a product automaton between $TST_\theta$ and $TST_c$ that does not induce an exponential state explosion since $TST_\theta$ and $TST_c$ “align” in a suitable way due to the particular control laws in Section V-C. In Section V-B, we then present the reactive planning method that consists of a combination of a game-based approach and graph search techniques (boxes in the middle of Fig. 4).  

In Algorithm 3 presented below, we summarize the reactive planning algorithm that is presented in this section. In the remainder, we present and explain the steps of Algorithm 3. In line 1, abstract the ReSITL formula $\theta(M)$ into an MITL formula $\phi := Tr(\theta(M)) = \theta(AP)$. Note that we abstract $\theta(M)$, which depends on deterministic predicates and uncontrollable propositions $M$ (recall that $M := M^{det} \cup M^{uc}$), as opposed to $\phi(M)$ in Section III-A by the transformation $Tr(\cdot)$. Based on $\phi$, construct $TST_c := (S, s_0, \Lambda, \Gamma, c, t, \Delta, \lambda, \gamma, A)$. According to Section II-B (Line 2 in Algorithm 3). We again assume that uncontrollable propositions $p_i \in AP$, i.e., $p_i$ with $Tr^{-1}(p_i) \in M \cap M^{uc}$, are modeled as in Fig. 3. In Line 3, perform operations [O1] and [O2] on $TST_c$ to obtain the timed signal transducer $TST_\theta := (S^\theta, s_0, \Lambda, \Gamma, c, t, \Delta^\theta, \lambda, \gamma, A^\theta)$. Note that checking [O1] and [O2] is computationally tractable if Assumption 1 holds due to the determination in Section IV.

A. Timed Abstraction of the Dynamical Control System

In line 4 of Algorithm 3, we abstract the system in (5) into a timed signal transducer $TST_m := (S, \tilde{S}_0, \tilde{\Lambda}, \tilde{c}, \tilde{\Delta}, \tilde{\lambda})$ (top left box in Fig. 4). Note the absence of output labels, invariants, and a Büchi acceptance condition, and that $\tilde{c}$ is a scalar. The transition relation $\tilde{\Delta}$ is now based on the ability of the system to switch in finite time, by means of a feedback control law $u_{\tilde{\theta}}(x,t)$ between elements in $Tr^{-1}(BC(TST_m)) \subseteq BC(\tilde{\Lambda})$, where $\tilde{\Lambda} := M$ and $BC(TST_m) := \{ z \in BC(AP) \mid \exists s \in S^\theta, \lambda(s) = \tilde{s} \}$. It is assumed that a library of such logic-based feedback control laws $u_{\tilde{\theta}}(x,t)$ is available, e.g., as presented in Section V-C. Assume that $|S| = |Tr^{-1}(BC(TST_m))|$ and let $\tilde{\lambda} : S \rightarrow Tr^{-1}(BC(TST_m))$, where for $\tilde{s}, \tilde{s}' \in \tilde{S}$ with $\tilde{s} \neq \tilde{s}'$, it holds that $\tilde{\lambda}(\tilde{s}) \neq \tilde{\lambda}(\tilde{s}')$ so that each state is uniquely labeled by $\tilde{\lambda}$, i.e., each state indicates exactly one Boolean formula from $Tr^{-1}(BC(TST_m))$. Note that $TST_\theta$ and $TST_m$ bow “align” in a way that will allow to avoid a state space explosion when forming a product automaton between them. A transition from $s \rightarrow s'$ is indicated by $(\tilde{s}, \tilde{g}, 0, \tilde{s}') \in \tilde{\Delta}$ where $\tilde{g}$ is a guard that depends on $\tilde{S}$. In particular, we assume that $\tilde{g}$ encodes intervals of the form $(w, r, w', r')$, $C(w), C(r), C(w'), C(r')$, or conjunctions of them, where $C(w), C(r) \in \mathbb{Q} \subseteq \mathbb{C}$.

Definition 8 (Transitions in $TST_m$): There exists a transition $\tilde{\delta} := (s, g, 0, s') \in \tilde{\Delta}$ if, for all $\tau > 0$ with $\tau = \tilde{g}$ and for all $x_0 \in \mathbb{R}^n$ with $(x_0, s^+, \tilde{\mu}) = \tilde{\lambda}(s)$, there exists a control law $u_{\tilde{\theta}}(x,t)$ so that the solution $x(t)$ to (5) is such that  

1) either, for all $t \in [0, \tau]$, $(x(t), s^+, \tilde{\mu}) = \tilde{\lambda}(s)$ and $(x(\tau), s^+, \tilde{\mu}) = \tilde{\lambda}(\tilde{s})$;  
2) or, for all $t \in [0, \tau]$, $(x(t), s^+, \tilde{\mu}) = \tilde{\lambda}(s)$ and there exists $\tau' > \tau$ such that, for all $t \in (\tau, \tau', (x(t), s^+, \tilde{\mu}) = \tilde{\lambda}(\tilde{s})$;  

for which we define $\tilde{\lambda}(\tilde{\delta}) := \tilde{\lambda}(s')$ in the former and $\tilde{\lambda}(\tilde{\delta}) := \tilde{\lambda}(\tilde{s})$ in the latter case.

The two types of transitions in the above definition can be thought of as transitioning into closed and open regions in $\mathbb{R}^n$, respectively. Note that such a control law $u_{\tilde{\theta}}(x,t)$ has to ensure invariance and finite-time reachability properties. Note also that $s^+$ is used in Definition 8 since controlled transitions will only happen when all uncontrollable propositions are false. Finally, the set $\tilde{S}_0$ consists of all elements $\tilde{s}_0 \in \tilde{S}$, such that $(x_0, s^+, \tilde{\mu}) = \tilde{\lambda}(\tilde{s}_0)$.

According to line 5 of Algorithm 3, we next form a product automaton $TST_m^{\theta}$ (bottom box in Fig. 4) of $TST_\theta$ and $TST_m$ that avoids a state space explosion that is typically the outcome of forming automata products. This follows since each input label of a state or transition in $TST_\theta$ corresponds to one state label of $TST_m$, i.e., $TST_m$ and $TST_m^{\theta}$ align in a way, so that $TST_m^{\theta}$ (defined below and corresponding to the product of $TST_\theta$ and $TST_m$) has no more states than $TST_\theta$. Our approach relies on: 1) The removal of transitions from $TST_\theta$, and 2) constraining guards $g$ of transitions in $TST_\theta$ to account for guards $\tilde{g}$ in $TST_m$. Let us, without loss of generality, assume that each input label of a transition in $TST_\theta$ contains every literal from $\tilde{M}$ and does not contain any disjunctions.\footnote{For each transition $\delta := (s, g, r, s') \in \Delta^\theta$ for which there exists $x \in \mathbb{R}^n$, such that $(x, s^+, \tilde{\mu}) = Tr^{-1}(\tilde{\lambda}(\tilde{s}))$, remove $\delta$ if a) there exists no transition $\delta := (\tilde{s}, \tilde{g}, 0, \tilde{s}') \in \tilde{\Delta}$ with $\tilde{\lambda}(s) = Tr(\tilde{\lambda}(\tilde{s}))$, and $\tilde{\lambda}(\tilde{s}') = Tr(\tilde{\lambda}(\tilde{s}'))$,\footnote{Note that each input label of a transition in $TST_\theta$ can be converted into full disjunctive normal form. Then, this transition can be split into several transitions, one for each disjunct, where each new input label corresponds to exactly one of the disjuncts.}}

[O3] For each transition $\delta := (s, g, r, s') \in \Delta^\theta$ for which there exists $x \in \mathbb{R}^n$, such that $(x, s^+, \tilde{\mu}) = Tr^{-1}(\tilde{\lambda}(\tilde{s}))$, remove $\delta$ if a) there exists no transition $\delta := (\tilde{s}, \tilde{g}, 0, \tilde{s}') \in \tilde{\Delta}$ with $\tilde{\lambda}(s) = Tr(\tilde{\lambda}(\tilde{s}))$, and $\tilde{\lambda}(\tilde{s}') = Tr(\tilde{\lambda}(\tilde{s}'))$,\footnote{Note that each input label of a transition in $TST_\theta$ can be converted into full disjunctive normal form. Then, this transition can be split into several transitions, one for each disjunct, where each new input label corresponds to exactly one of the disjuncts.}
and for which \((x, s^+, \mu) \models T^{-1}(\lambda(\delta))\) implies \((x, s^+, \mu) \models T^{-1}(\lambda(\delta))\).

Remove the corresponding \(\delta\) from \(A^0\).

We follow two goals with operation [O3]. First, we only consider to remove transitions that are induced by uncontrollable propositions being false, i.e., when \(s = s^+\). This is important as we would like to keep transitions with \(s \neq s^+\) for the reactive planning. Note in particular that, if there exists \(x \in \mathbb{R}^n\), such that \((x, s^+, \mu) \models T^{-1}(\lambda(\delta))\), then there exists \(x \in \mathbb{R}^n\), such that \((x, s, \mu) \models T^{-1}(\lambda(\delta))\) for \(s \neq s^+\). Second, we remove such transitions if there exists no control \(u_3\) that can simulate the transition in the system (5).

[O4] For each transition \(\delta_0 := (s_0, g, r, s') \in \Delta\), remove \(\delta_0\) if \((x_0, s^+, \mu) \neq T^{-1}(\lambda(s'))\) or if there exists no \(s \in \mathbb{B}|\Delta^m|\), such that \((x_0, s, \mu) \neq T^{-1}(\lambda(\delta_0))\). Remove the corresponding \(\delta_0\) from \(A^0\).

Operation [O4] takes care of the initial condition \(x_0\). If \(s_0\) is infeasible the previous initial condition \(x_0\).

Denote next the obtained sets by \(S^m, \Delta^m, \text{ and } A^m\) for which \(S^m \subseteq \mathbb{B}^m, \Delta^m \subseteq \Delta^m\) and \(A^m \subseteq \mathbb{B}^m\). We further take care of the timings including an additional clock into \(\text{TST}_\theta\). Therefore, let \(e^m := [e^r, e^\mu]^2\) and perform the operation.

[O5] For each transition \(\delta^m := (s, g, r, s') \in \Delta^m\) for which there exists \(x \in \mathbb{R}^n\), such that \((x, s^+, \mu) \models T^{-1}(\lambda(s^m))\), let \(g' = g \iff \delta = (s, g, r, s') \in \Delta\) with \(\lambda(s) = T\lambda(\delta(s))\), \(\lambda(s') = T\lambda(\delta(s'))\), and for which \((x, s^+, \mu) \models T^{-1}(\lambda(\delta))\) implies \((x, s^+, \mu) \models T^{-1}(\lambda(\delta))\). Replace \(g\) and \(r\) in \(\delta^m\) with \(g'\) and \(r\), respectively, where \(r^m\) is obtained in an obvious manner.

We emphasize that adding \(\bar{e}\) and \(\bar{g}\) is crucial to ensure correctness. Let the modified timed signal transducer be denoted by \(\text{TST}_\theta^m := (S^m, \Delta^m, \Delta^m, \text{ and } A^m)\) and note that \(L(\text{TST}_\theta^m) \subseteq L(\text{TST}_\theta)\).

Remark 5: The operations [O3]–[O5] result in the timed signal transducer \(\text{TST}_\theta^m\) that, by construction, restricts the behavior of \(\text{TST}_\theta\) exactly to the behavior allowed by \(\text{TST}_\theta^m\) and corresponds hence to a product automaton without exhibiting an exponential state space explosion.

B. Reactive Plan Synthesis

Based on \(\text{TST}_\theta^m\), let us now present the reactive planning method depicted in the boxes in the middle of Fig. 4. We first derive a nominal plan \(\nu_0 : \mathbb{R}_{\geq 0} \to BC(M)\) from \(\text{TST}_\theta^m\) based on the assumption that \(s(t) = s^+\) for all \(t \in \mathbb{R}_{\geq 0}\). This plan is executed until \(s(t_{\text{replan}}) \neq s^+\) for some \(t_{\text{replan}} \in \mathbb{R}_{\geq 0}\), the moment when reactive and online replanning is needed. In line 6 of Algorithm 3, let \(\text{TST}_\theta^m\) again be modified to not exhibit Zeno behavior and let \(\text{RA}_C(\text{TST}_\theta^m) := (\tilde{Q}, \tilde{q}_0, \tilde{A}_R, \tilde{\lambda}_R)\) be the region automaton of \(\text{TST}_\theta^m\) based on \((S^m \times \mathbb{R}_{\geq 0}, \Rightarrow)\) and Definitions 6 and 7 (lines 7 and 8 of Algorithm 3).\(^{10}\) Replanning may now require to take, at an unknown time instant \(t_{\text{replan}}\), a transition that is not contained within the nominal plan. Those instances may possibly require an infeasible discontinuity in the physical state \(x\) that we need to rule out.

Example 3: To illustrate the aforementioned issue, consider Fig. 5. For the top left state, there exist two transitions to the top right and the bottom left state. Assume the former transition can be realized by the control law \(u_3(x, t)\). Starting from the top left state, the initial plan will consider the transition with \(s = 1\) to the top right state implying that \(u_3(x, t)\) is used until time \(t = t_{\text{replan}}\), such that \(0 < x(t_{\text{replan}}) < c\). After replanning, however, the other transition with \(s = \top\) to the bottom left state has to instantaneously be taken requiring to immediately achieve \(x \leq -c\). Such a discontinuity in \(x(t)\) is not realizable in (5) that only admits continuous \(x(t)\).

One way of dealing with this issue is to modify the predecessor operator. Recall therefore that a state \(\tilde{q} \in \tilde{Q}\) in \(\text{RA}_C(\text{TST}_\theta^m)\) consists of the elements \(\tilde{q} := (s, a, i) \in L(\text{TST}_\theta^m) \times A \times \{1, \ldots, |A^m|\}\) and redefine now \(\pi(W)\) to

\[
\pi(W) := \{\tilde{q} \in \tilde{Q} \mid s \in S^m, a \in A, i \in \{1, \ldots, |A^m|\}, (x, s, \mu) \models T^{-1}(\lambda(s))\},
\]

for \(s \in \mathbb{R}^n\), such that \((x, s, \mu) \models T^{-1}(\lambda(\delta))\).

The second condition in \(\pi(W)\) now additionally ensures that all \(x\) that satisfy the state label of \(s\) also satisfy the state labels of the transition \(\delta\) as well as the next state \(s'\). As a consequence, an instantaneous transition from \(\tilde{q}\) to \(\tilde{q}'\) due to \(s(t) \neq s^+\) can happen without requiring that \(x(t)\) is discontinuous. We emphasize, again, that this condition is necessary with respect to the solutions to (5). Let \(W\) be obtained from Algorithm 2 with \(\text{RA}_C(\text{TST}_\theta^m)\) and \(\pi : 2^\tilde{Q} \to 2^\mathbb{R}\) as the input (line 9 of Algorithm 3).

Initial Plan Synthesis: For line 10 in Algorithm 3, let \(d_p(t)\), as opposed to Section III-B, now be obtained from \(\text{RA}_C(\text{TST}_\theta^m)\) as follows. We find, using graph search techniques, a sequence \(q := (\tilde{q}_0, \tilde{q}_1, \ldots) := (q_0, q_{\tilde{m}})\) satisfying the Büchi acceptance condition \(\tilde{A}_R\) with \(\tilde{q}_j \in \tilde{Q} \cap W\) for each \(j \in \mathbb{N}\) and where \((\tilde{q}_j, \tilde{\delta}_{ij}, \tilde{q}_{ij+1}) \in \tilde{A}_R\) so that, for each \(\tilde{\delta}_{ij}\), there exists \(x \in \mathbb{R}^n\), such that \((x, s^+, \mu) \models T^{-1}(\delta_{ij})\). Additionally and for the initial transition \(\delta_0\), we again require that \(\gamma(\delta_0) = y\) to indicate \((x, s, \mu, 0) \models \theta\). Note in particular the restriction to \(\tilde{q}_j \in \tilde{Q} \cap W\) which will allow to replan if \(s(t_{\text{replan}}) \neq s^+\) for some \(t_{\text{replan}} \in \mathbb{R}_{\geq 0}\). We again find timings \(\tilde{t} := (\tilde{t}_0, \tilde{t}_1, \ldots) := (\tilde{t}_p, \tilde{t}_{\tilde{m}})\) that are associated with \(q\). Such a plan \(d_p(t)\) is guaranteed to exist if the conditions in Theorem 2 are satisfied. Recall that \(T_j := \sum_{k=0}^j \tilde{t}_j\), and define

\[
d_p(t) := \begin{cases} \lambda(\tilde{\delta}_{ij}) & \text{if } t = T_j \\ \lambda(s_j) & \text{if } T_j < t < T_{j+1}. \end{cases}
\]
We can now define the control law \( u(x, s, t) \) based on the plan \( d_{\mu}(t) \). Recall therefore that each transition \( \delta_{i,j} \) is associated, when projecting back to \( TST_S \), with a control law \( u_{\delta_{i,j}}(x, t) \) as explained in Section V-A. Recall the definition of \( T_{j} \) and let

\[
u(x, s, t) := \begin{cases} u_{\delta_{i}}(x, t) & \text{for } t \in [0, T_{j}) \\ u_{\delta_{j+1}}(x, t - T_{j}) & \text{for } t \in (T_{j}, T_{j+1}) \end{cases}
\]

and, for \( t = T_{j} \) with \( j \geq 2 \), let

\[
u(x, s, T_{j}) := \begin{cases} u_{\delta_{j+1}}(x, 0) & \text{if } \lambda(\delta_{j+1}) = d_{\mu}(T_{j}) \\ u_{\delta_{j}}(x, T_{j}) & \text{if } \lambda(\delta_{j}) = d_{\mu}(T_{j}). \end{cases}
\]

Note that \( u(x, s, T_{j}) \) in particular accounts for the two types of transitions in Definition 8.

**Corollary 3**: Assume that \( s(t) = s^+ \) for all \( t \in \mathbb{R}_{>0}, \tau \in W \) and there exists \((\tau_{\theta}, \delta_{0}, \tau_{\theta}') \in \Delta_{R} \) with \( \gamma(\delta_{0}) = y \), then \( d_{\mu}(t) \) as in (7) exists and \( u(x, s, t) \) results in \((x, s, \mu, 0) = \theta \).

**Proof**: Similar to Theorem 2 and by the construction of TST_{\theta}, it follows that \( \theta \) is satisfiable given that \( \tau_{\theta} \in W \) and that there exists \((\tau_{\theta}, \delta_{0}, \tau_{\theta}') \in \Delta_{R} \) with \( \gamma(\delta_{0}) = y \). It directly follows that, in this case, a plan \( d_{\mu}(t) \) exists. Note next that by construction of TST_{\theta} and TST_{\theta}', each transition \( \delta \) in TST_{\theta}' can be realized in (5) by an associated control law \( u_{\delta}(x, t) \). By the construction of the plan \( d_{\mu}(t) \) and the associated control law \( u(x, t) \), it follows trivially that \( u(x, s, t) \), build from a sequence of such \( u_{\delta}(x, t) \), results in \((x, s, \mu, 0) = \theta \).

**Reactive and Online Replanning**: If hence \( s(t) = s^+ \) for all \( t \in \mathbb{R}_{>0} \), there is nothing left to do and we apply \( u(x, s, t) \) as in line 14 of Algorithm 3. Assume that, at time \( t_{\text{replan}} \), the system is in state \( \tau_{\theta} \).

We then find an updated sequence \( q_{\text{replan}} := (\tau_{j}', \tau_{j+1}', \ldots) \) satisfying the Büchi acceptance condition \( \Delta_{R} \) again with \( \tau_{j} \in W \) for each \( j > j' \) and where \((\tau_{j}, \delta_{j}, \tau_{j+1}) \in \Delta_{R} \) so that \( 1 \langle x(t_{\text{replan}}), s(t_{\text{replan}}), \mu \rangle \rangle \) \( T_{R}^{-1}(\delta_{j}), \mu \rangle \rangle \) and 2) for each \( \delta_{j} \) with \( j > j' \) there exists \( x \in \mathbb{R}^{n} \), such that \((x, s^+, \mu) \rangle \rangle \) \( T_{R}^{-1}(\delta_{j}). \)

If \( t_{\text{replan}} = 0 \), it is additionally required that \( \gamma(\delta_{j}) = y \). We then find timings \( \tau := (\tau_{j}', \tau_{j+1}', \ldots) \) that are associated with \( q_{\text{replan}} \) based on this updated sequence, we recalculate \( d_{\mu}(t) \) in (7) and \( u(x, s, t) \) in an obvious manner.

**Theorem 4**: Assume that \( s(t) \) is according to Assumption 2, \( \tau_{\theta} \in W \), and there exists \((\tau_{\theta}, \delta_{0}, \tau_{\theta}') \in \Delta_{R} \) with \( \gamma(\delta_{0}) = y \), then finding an initial plan \( d_{\mu}(t) \) and updating \( u(x, s, t) \) in the previously described manner in case that \( s(t_{\text{replan}}) \neq s^+ \) results in \((x, s, \mu, t) \rangle \rangle \) \( \theta \).

**Proof**: The assumptions that \( \tau_{\theta} \in W \) and that there exists \((\tau_{\theta}, \delta_{0}, \tau_{\theta}') \in \Delta_{R} \) with \( \gamma(\delta_{0}) = y \), again guarantee that there exists an initial plan \( d_{\mu}(t) \). Due to the properties of \( W \) and given that \( s(t) \) is according to Assumption 2, it holds that a new plan and an updated \( u(x, s, t) \) can always be found whenever \( s(t_{\text{replan}}) \neq s^+ \). Each such instantaneous transition is well defined in the sense of not requiring a discontinuity in \( x(t) \) due to the modified definition of \( \gamma(W) \).

We remark that Assumption 2 is not only necessary for the game-based approach in Algorithm 2 but that the assumption is also necessary to be able to replan. Without Assumption 2, there is no information about the value of \( s(t) \) shortly after \( t_{\text{replan}} \). By Assumption 2, there follows an open time interval in which

\[ s(t) = s^+ \text{ after } t_{\text{replan}} \text{ so that a next state can be selected whose state label is satisfied by } s^+. \]

Further note that Assumption 2 effectively poses an upper bound on the frequency of times that replanning is initiated.

To conclude this section, we note that a combination of graph search techniques and a game-based approach has been presented. The game-based approach ensures that it is always possible to make progress toward satisfying the Büchi acceptance condition by ruling out "bad" transitions, while graph search techniques actually enforce this progress.

**C. Feedback Control Under STL Specifications**

In this section, we discuss the control laws \( u_{\delta}(x, t) \) that are supposed to achieve the transitions \( \delta := (\bar{s}, \bar{y}, 0, \bar{s}') \in \Delta \) in Definition 8 for the timed abstraction TST_{\theta}. In particular, such transitions can be captured by the STL formulas

\[ G[0,\tau]_{\mu_{\text{inv}}}(x) \land F_{\tau} \mu_{\text{reach}}(x) \land G[0,\tau]_{\mu_{\text{ws}}}(x) \]

\[ G[0,\tau]_{\mu_{\text{inv}}}(x) \land G[0,\tau]_{\mu_{\text{reach}}}(x) \land G[0,\tau]_{\mu_{\text{ws}}}(x) \]

where \( \mu_{\text{inv}}(x) := \lambda(\bar{s}) \) and \( \mu_{\text{reach}}(x) := \lambda(\bar{s}') \) are deterministic predicates as in (3), and where \( \tau \in \mathbb{R}_{>0} \) with \( \tau = \gamma \), while \( \mu_{\text{ws}}(x) \) encodes a compact set \( \mathcal{B} \) according to Section IV; \( \mathcal{B} \) can be any compact set, typically the workspace. With \( \mu_{\text{inv}}(x), \mu_{\text{reach}}(x), \mu_{\text{ws}}(x) \), we can now associate predicate functions \( h_{\text{inv}}(x), h_{\text{reach}}(x), h_{\text{ws}}(x) \).

There is a plethora of recent works that have addressed the problem of controlling systems as in (5) under spatio-temporal constraints as in (8) or (9). In particular [4] and [20] address the control problem by time-varying control-barrier functions and fixed time control Lyapunov functions, respectively. For robotic specific problem setups, funnel control laws to solve (8) or (9) have also appeared in [18], while optimization-based methods are presented in [19]. Another approach, relying on time-varying vector fields, has appeared in [24]. We are, purposefully and with respect to page limitations, not presenting a specific type of feedback control law here and emphasize that our proposed reactive planning method is agnostic to feedback control laws that can achieve the STL specification as in (8) or (9). Note that the previously mentioned works pose certain assumptions on the systems dynamics in (5) as well as on the form of \( h_{\text{inv}}(x) \), \( h_{\text{reach}}(x) \), and \( h_{\text{ws}}(x) \).

We remark that controlling systems under timed specifications of the type in (8) or (9) has recently attracted interest in the research community so that we expect more progress in this respect.

**VI. COMPLETENESS AND COMPLEXITY**

In summary, the presented framework consists of: 1) Translating the ReRiSITL specification \( \phi \) into a ReSITL specification \( \theta \) in Section IV, and 2) reactive planning under this ReSITL specifications \( \theta \) in Section V, as summarized in Algorithm 3. The framework is sound in the sense of Theorems 3 and 4, but not necessarily complete, i.e., there may exist a solution even though we may not find it. There are three reasons for such conservatism. First, the translation from the ReRiSITL specification \( \phi \) to the ReSITL specification \( \theta \) may induce conservatism as discussed in Section IV. Second, in line 6 of Algorithm 3, we need to modify TST_{\theta} to avoid Zeno behavior. This operation potentially induces conservatism that can, however, be reduced as also discussed previously. Third, the construction of nonlinear control laws,
presented in Section V-C, may introduce conservatism. This is inherent in nonlinear control and we do not view this as a drawback of our method.

The presented framework consists of several computationally expensive operations. Fortunately, these operations can be performed offline. We focus on space complexity. First, the translation from the MITL formula \( \varphi \) to the timed signal transducer \( \text{TST}_\varphi \) induces \( O(|\varphi|) \) clocks and \( 2O(|\varphi|M) \) states, where \( |\varphi| \) denotes the complexity of \( \varphi \) and \( M \) is related to the length of the maximum time interval in \( \varphi \) (see [32, Th. 6.7]). Operations \([O1] \) and \([O2] \), which transform \( \text{TST}_\varphi \) into \( \text{TST}_\rho \), ease the complexity by removing a considerable number of states and transitions from \( \text{TST}_\varphi \). An exact number is in general not quantifiable as those removals depend on predicate dependencies in the specification \( \theta \). Operations \([O3] \) and \([O4] \) further reduce states and transitions from \( \text{TST}_\rho \) to obtain the product automaton \( \text{TST}_\rho^p \). Note that we obtain computational benefits over existing methods that would induce additional \( O(|S^m||\mathcal{S}|) \) states. The operation \( R\mathcal{A}_C(\text{TST}_\rho^p) \) results in an automaton with \( O(|S^m||\text{len}(c_m)|) \) states, where \( \text{len}(c_m) \) denotes the length of clock constraints in \( \text{TST}_\rho^p \) (see [29, Sec. 4.3]). The translation from \( R\mathcal{A}_C(\text{TST}_\rho^p) \) to \( \mathcal{T}\mathcal{A}_C(\text{TST}_\rho^p) \) results in an automaton with \( O(|Q|\mathcal{|A}_\mu|) \) states, which can considerably be reduced as discussed in Remark 4. The time complexity of Algorithm 2 and graph search techniques to find a plan \( d_\mu(t) \) follows standard arguments. Operations \([O1] \), \([O2] \), \([O3] \), and \([O4] \) involve solving nonlinear integer programs, and in particular mixed integer linear programs when Assumption 1 holds.

### VII. Simulations

We consider a unicycle model with dynamics \( \dot{z} = f(z) + g(z)u \) and state \( z := [x^T \; \; x_u]^{T} := [x_s \; \; x_y \; \; x_a]^{T} \) to model the two-dimensional position and orientation, respectively. Here, \( u := [v \; \omega]^T \) contains the translational and rotational control inputs. In particular, let \( f(x) := 0.5 \cdot [-\text{sat}(x_s) \; \; -\text{sat}(x_y) \; \; 0]^T \), where \( \text{sat}(x) = x \) if \( |x| \leq 1 \) and \( \text{sat}(x) = 1 \) otherwise, and let \( g(x) := \begin{bmatrix} \cos(x_a) & 0 & 0 \\ \sin(x_a) & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \). To obtain \( u(z, t) \), we use here the time-varying control barrier functions from [4]. In particular, time-varying control barrier functions adapted for nonlinear systems from [45] are used for which no knowledge of \( f(z) \) is required.

For this system, the imposed ReSiSTL specification \( \phi \) is the one given in Example 1. The specification \( \phi \) is rich enough to illustrate all theoretical findings (i.e., how to deal with risk predicates, uncontrollable propositions, and past temporal operators) and yet basic enough to explain all subtleties of \( \phi \) and the reactive and risk-aware control synthesis.

Recall the determination of risk predicates according to Section IV in Example 2 resulting in the ReSiSTL specification \( \theta := Tr(\phi) = F(0,5) \mu_{R1}^{\text{det}} \land G(0,\infty) (\mu_{G1}^{\text{det}} \land \mu_{G2}^{\text{det}} \land (\mu_{F1}^{\text{det}} \land \mu_{F2}^{\text{det}})) \) for which initially \( (x(0), s^x, \mu) \models \neg \mu_{R1}^{\text{det}} \land \mu_{G1}^{\text{det}} \land \mu_{G2}^{\text{det}} \land \neg \mu_{F2}^{\text{det}} \) is assumed. For the construction of \( \text{TST}_\phi \) in Section V-A, we assume that we have control laws \( u_{\phi}(x, t) \) that can accomplish each transition \( \delta \) as per Definition 8 with \( \hat{g} := [1, \infty) \).

### Fig. 6. Unicycle model for the ReSiSTL specification \( \phi \) with \( \zeta := 5 \) and when an uncontrollable event occurs.

**Setting 1:** With respect to Assumption 2, we first assume that \( \zeta := 1 \). Recall that \( \zeta \) determines the frequency by which the uncontrollable event \( \mu_{uc} \) may occur. In this case, the set \( W \) does not contain the element \( \eta_0 \), i.e., \( \eta_0 \notin W \), so that by Theorem 4 no plan \( d_\mu(t) \) is found that satisfies \( \theta \) and consequently \( \phi \). Note that this follows mainly since \( s(t) = \text{proj}_{\mu_{uc}}(s(t)) = \top \) may occur within \( \zeta \) time unit intervals implying that, in the worst case, \( \mu_{R2} \) should always be true so that there is no time to satisfy \( \mu_{R1} \).

**Setting 2:** By increasing \( \zeta \), the frequency by which the uncontrollable event \( \mu_{uc} \) may occur is decreased. We set \( \zeta := 5 \) and now observe that \( \eta_0 \in W \). The synthesized initial plan \( d_\mu(t) \) is as follows:

\[
\begin{align*}
\mu_{R1}^{\text{det}} \land \mu_{G1}^{\text{det}} \land \mu_{G2}^{\text{det}} \land \neg \mu_{R2}^{\text{det}} \land \neg \mu_{uc} \quad &\quad t \in (0, 4) \\
\mu_{R1}^{\text{det}} \land \mu_{G1}^{\text{det}} \land \mu_{G2}^{\text{det}} \land \neg \mu_{R2}^{\text{det}} \land \neg \mu_{uc} \quad &\quad t \in [4, 5, 7] \\
\mu_{R1}^{\text{det}} \land \mu_{G1}^{\text{det}} \land \mu_{G2}^{\text{det}} \land \neg \mu_{uc} \quad &\quad t \in (5, 7, \infty).
\end{align*}
\]

However, now assume that \( s(1) = \text{proj}_{\mu_{uc}}(1) = \top \) so that at \( t_{\text{replan}} = 1 \) replanning is needed. Our revised plan then is

\[
\begin{align*}
\neg \mu_{R1}^{\text{det}} \land \mu_{G1}^{\text{det}} \land \mu_{G2}^{\text{det}} \land \neg \mu_{R2}^{\text{det}} \land \neg \mu_{uc} \quad &\quad t \in (0, 1) \\
\neg \mu_{R1}^{\text{det}} \land \mu_{G1}^{\text{det}} \land \mu_{G2}^{\text{det}} \land \neg \mu_{R2}^{\text{det}} \land \mu_{uc} \quad &\quad t = 1 \\

\neg \mu_{R1}^{\text{det}} \land \mu_{G1}^{\text{det}} \land \mu_{G2}^{\text{det}} \land \neg \mu_{uc} \quad &\quad t \in (1, 2) \\
\neg \mu_{R1}^{\text{det}} \land \mu_{G1}^{\text{det}} \land \mu_{G2}^{\text{det}} \land \mu_{uc} \quad &\quad t \in [2, 3] \\
\neg \mu_{R1}^{\text{det}} \land \mu_{G1}^{\text{det}} \land \mu_{G2}^{\text{det}} \land \neg \mu_{uc} \quad &\quad t \in [3, 4) \\
\neg \mu_{R1}^{\text{det}} \land \mu_{G1}^{\text{det}} \land \mu_{G2}^{\text{det}} \land \mu_{uc} \quad &\quad t \in [4, \infty).
\end{align*}
\]

i.e., to prepone satisfying \( \mu_{R1}^{\text{det}} \) and to satisfy \( \mu_{R2}^{\text{det}} \) right after and within 3 time units from when \( \mu_{uc} \) happened. The simulation results for this case are depicted in Fig. 6.

Simulations were performed on a 1.4-GHz quad-core Intel Core i5 with 8 GB RAM. Construction of \( \text{TST}_\theta \) and \( \mathcal{T}\mathcal{A}_C(\text{TST}_\theta) \) took 2.5 and 78.5 s, respectively, while Algorithm 2 and the graph search took 130 and 15.5 s, respectively. All MATLAB implementations and an animation of the simulation can be found in [52].

### VIII. Conclusion

This article has presented ReSiSTIL as a significant extension of STL. ReSiSTIL additionally allows to consider the risk of not satisfying an ReSiSTIL specification as well as allowing to consider environmental events such as sensor failures. We
have then proposed an algorithm to check if such an ReRiSITL specification is satisfiable. Lastly, we have proposed a reactive planning and control framework for dynamical systems under ReRiSITL specifications by combining a game-based approach with graph search techniques.

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REFERENCES


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