On the Structural Target Controllability of Undirected Networks

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***Abstract*—In this article, we study the *target controllability problem* of networked dynamical systems, in which we are tasked to steer a subset of network nodes toward a desired objective. More specifically, we derive necessary and sufficient conditions for the *structural target controllability* of linear time-invariant (LTI) systems with symmetric state matrices, such as those representing undirected dynamical networks with unknown link weights. To achieve our goal, we first characterize the generic rank of *symmetrically structured matrices*, as well as the modes of any numerical realization. Subsequently, we provide graph-theoretic necessary and sufficient conditions for the structural target controllability of undirected networks with multiple control nodes. In addition, we show that these results can be extended and lead to a necessary and sufficient condition of the structural output controllability. However, different from structural target controllability, we prove that verifying the proposed conditions on structural output controllability in undirected networks is NP-hard.**

***Index Terms*—Networked control systems, structured linear systems, target controllability.**

I. Introduction

Complex networks are a powerful tool for modeling dynamical systems [1]–[3]. In particular, when analyzing and designing networked dynamical systems, it is crucial to verify their controllability, i.e., the existence of an input sequence allowing us to drive the states of the system toward arbitrary states within finite time. Nonetheless, verifying such a property requires full knowledge of the parameters describing the system's dynamics [4]. However, in many applications involving large-scale networks, those parameters are difficult, or even impossible, to obtain [3]. Alternatively, it is practically more viable to identify the existence or absence of dynamical interconnections among the states of a network, without characterizing the strength of the interactions. Subsequently, it is of interest to infer system properties, such as controllability, using exclusively information about the system structure and tools from graph theory [5].

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Seminal work on a graph-theoretic analysis of controllability can be found in [6], in which the notion of *structural controllability* was introduced. Following this work, in [7]–[10], necessary and sufficient conditions for structural controllability of multi-input linear time-invariant (LTI) systems using various graph-theoretic notions were provided.

Nonetheless, existing results on structural controllability assumed implicitly that the parameters are either fixed zeros or independent free variables. Such an assumption is often violated in practical scenarios, for instance, when the system is characterized by undirected networks [11], or when different interconnections in the system are strongly correlated [12]. Consequently, it is of interest to provide necessary and sufficient conditions for structural systems characterized by graphs with special weight constraints, such as those considered in [13] and [14]. However, the result in [13] is not applicable to systems modeled by undirected graph, whereas the approach in [14] may suffer from scalability issues in large-scale systems.

Recently, Menara *et al.* [15] and Mousavi *et al.* [16] proposed graph-theoretic necessary and sufficient conditions for structural controllability of dynamical systems modeled by a *symmetric* graph. Different from their approaches, in this article, we provide a full characterization of the controllable modes using structural information of an undirected network via tools from algebraic geometry and graph theory, which facilitates a deeper understanding of structural controllability for systems involving symmetric parameter constraints.

Nonetheless, in certain scenarios, it suffices to steer a subset of states toward desired values, instead of the full set of states [17]. Given a subset of states, the ability to steer this subset of states arbitrarily is termed *target controllability* [17], [18]. The target controllability problem is a particular case of the *output controllability* problem [19], where we aim to steer the outputs of the system. Although graph-theoretic conditions on the *strong target controllability*, a stronger notion of target controllability, are proposed in [20] and [21], it is known that there are no necessary and sufficient conditions of structural target controllability and structural output controllability—see [18] and [22] for details.

In this article, we provide necessary and sufficient conditions for target and output controllability for undirected networks, which extend the results in a preliminary version of our work [23]. Furthermore, we provide a computational complexity analysis of assessing structural output controllability for general linear systems involving symmetric state matrices. In summary, the main contributions of the article are fourfold: We first provide full characterizations of the generic spectral properties of symmetrically structured systems. Leveraging those generic properties, we then show that the symmetry of the state matrix allows us to generalize the PBH test to characterize structural target controllability, which in turn enables us to provide graph-theoretic necessary and sufficient conditions for structural target controllability. Subsequently, we show that those results can be extended to a necessary and sufficient condition for structural output controllability. Finally, we explore the computation complexity of verifying these graph-theoretic conditions.

The rest of the article is organized as follows. In Section II, we formulate the problems under consideration. Some preliminaries in linear systems and graph theory are recalled in Section III. In Section IV, we derive main results, of which proofs are relegated to the Appendix. Illustrative examples are depicted in Section V. Finally, we conclude the article in Section VI.

II. Problem Statements

Consider an LTI system whose dynamics is captured by

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^k$, and $u \in \mathbb{R}^m$ are the state, output and input vectors, respectively. We refer to the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{k\times n}$ as the state, input, and output matrix, respectively. In this article, we consider the following assumption.

Assumption 1: The state matrix $A \in \mathbb{R}^{n\times n}$ is symmetric, i.e., $A = A^\top$.

This symmetry assumption is motivated by control problems arising in undirected networked dynamical systems [15], [16], [24]. Hereafter, we use the 3-tuple (A, B, C) to represent the system (1). In particular, we use the pair (A, B) to denote a system without a measured output.

When we aim to study the system properties in (A, B), in general, it is required to have access to the values of the entries in A and B [25]. However, in certain scenarios, only the presence/absence of interactions between inputs and states, or among states, are available. In other words, only the sparsity patterns of the matrices A and B are available. Consequently, we focus on studying the relationship between the sparsity pattern of the 3-tuple (A, B, C) and the controllability of the system. To do so, we first introduce few definitions from structural system theory.

Definition 1 (Structured and Symmetrically Structured Matrices): A matrix $\bar{M} \in \{0, \star\}^{n\times m}$ is called a *structured matrix*, if $[\bar{M}]_{ij}$ is either a fixed zero or an independent free parameter, typically denoted by $\star$. In particular, we define a matrix $\bar{M} \in \{0, \star\}^{n\times n}$ to be *symmetrically structured*, if the value of the free parameter associated with $[\bar{M}]_{ij}$ is constrained to be the same as the value of the free parameter associated with $[\bar{M}]_{ji}$ for all $j < i$.

Example 1: Consider the matrices

$$\bar{M} = \begin{bmatrix} 0 & m_{12} \\ m_{21} & 0 \end{bmatrix} \text{ and } \bar{A} = \begin{bmatrix} 0 & a_{12} \\ a_{12} & 0 \end{bmatrix}$$

where m_{12}, m_{21}, and a_{12} are independent parameters. In this case, $\bar{M}$ is a structured matrix, whereas $\bar{A}$ is symmetrically structured.

In addition, we refer to $\tilde{M}$ as a *numerical realization* of a (symmetrically) structured matrix $\bar{M}$, i.e., $\tilde{M}$ is a matrix obtained by independently assigning real numbers to each independent free parameter in $\bar{M}$.

Given a 3-tuple (A, B, C), we use $(\bar{A}, \bar{B}, \bar{C})$ to denote its structural counterpart; more specifically, $[\bar{A}]_{ij} = \star$ if $[A]_{ij} \neq 0$ and $[\bar{A}]_{ij} = 0$ otherwise. By Assumption 1, we assume that the structural matrix $\bar{A}$ is symmetrically structured. In this article, we are interested in the following system property.

Definition 2 (Structural Controllability [6]): A structural pair $(\bar{A}, \bar{B})$ is structurally controllable if there exists a numerical realization $(\tilde{A}, \tilde{B})$, such that the controllability matrix $Q(\tilde{A}, \tilde{B}) := [\tilde{B}, \tilde{A}\tilde{B}, \ldots, \tilde{A}^{n-1}\tilde{B}]$ has full row rank.

While controllability is concerned about the ability to steer all the states of a system, in certain cases, we are only interested in steering a subset of states. More specifically, given a set $\mathcal{T} \subseteq [n] := \{1, \cdots, n\}$, which we refer to as the *target set*, we are interested in steering the states indexed by the target set arbitrarily. This does not exclude the possibility of states in $[n] \setminus \mathcal{T}$ being steered as well. If given a system described by the pair (A, B), we are able to arbitrarily steer the states indexed by $\mathcal{T}$, we say that the pair (A, B) is *target controllable* with respect to $\mathcal{T}$ [17]. Furthermore, it is possible to consider a more general problem in which we are interested in steering the outputs, i.e., weighted combinations of system states of a system described by the 3-tuple (A, B, C). Similar to the definition of structural controllability, we can define *structural target controllability* and *structural output controllability* in the context of structured systems.

Definition 3 (Structural Output Controllability and Structural Target Controllability [18]): Given a target set $\mathcal{T} = \{i_1, \ldots, i_k\} \subseteq [n]$, define the *target matrix* $C_\mathcal{T} \in \mathbb{R}^{k\times n}$ by

$$[C_\mathcal{T}]_{\ell j} = \begin{cases} 1, & \text{if } j = i_\ell,\ i_\ell \in \mathcal{T} \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

The structural system $(\bar{A}, \bar{B}, \bar{C})$ is structurally output controllable if there exists a numerical realization $(\tilde{A}, \tilde{B}, \tilde{C})$ such that output controllability matrix $Q(\tilde{A}, \tilde{B}, \tilde{C}) := \tilde{C}[\tilde{B}, \tilde{A}\tilde{B}, \ldots, \tilde{A}^{n-1}\tilde{B}]$ has full row rank. Similarly, the structural pair $(\bar{A}, \bar{B})$ is structurally target controllable with respect to $\mathcal{T}$ if there exists a numerical realization $(\tilde{A}, \tilde{B})$, such that $Q(\tilde{A}, \tilde{B}, C_\mathcal{T})$ has full row rank.

Notice that structural target controllability is a special case of structural output controllability, provided that $\tilde{C}$ takes the particular form in (2). Thus, necessary and sufficient conditions for structural output controllability characterize structural (target) controllability. Hence, in this article, we seek to address the following problem.

Problem 1 (Structural Output Controllability Problem): Given a structured system $(\bar{A}, \bar{B}, \bar{C})$, where $\bar{A}$ is symmetrically structured, find necessary and sufficient conditions for $(\bar{A}, \bar{B}, \bar{C})$ to be structurally output controllable.

Additionally, provided such necessary and sufficient conditions exist, we would like to understand the computational complexity of the problem under consideration.

III. Notation and Preliminaries

In this section, we recall some useful concepts related to linear structural system theory and graph theory.

A. Structural System Theory

Consider a pair (A, B) whose dynamics is captured by (1). (A, B) is called *reducible* [9] if there exists a permutation matrix P, such that

$$PAP^{-1} = \begin{bmatrix} A_{11} & \mathbf{0} \\ A_{21} & A_{22} \end{bmatrix}, \quad PB = \begin{bmatrix} \mathbf{0} \\ B_2 \end{bmatrix} \tag{3}$$

where $A_{11} \in \mathbb{R}^{q\times q}$ and $B_2 \in \mathbb{R}^{(n-q)\times m}$, $1 \leq q < n$. The pair (A, B) is called *irreducible* otherwise.

Let λ be an eigenvalue of A and let $v \in \mathbb{R}^n$ be the associated eigenvector. By PBH test [25], an eigenpair (λ, v), which is also called a *mode* of the pair (A, B), is a *controllable mode* if $\text{rank}([\lambda I - A, B]) = n$, where the $[\cdot]$ denotes the matrix concatenation operator.

Consider a (symmetrically) structured matrix $\bar{M}$. Let $n_{\bar{M}}$ be the number of its independent $\star$-parameters and associate with $\bar{M}$ a parameter space embedded in $\mathbb{R}^{n_{\bar{M}}}$. Subsequently, we use a vector $\mathbf{p}_{\bar{M}} = (p_1, \ldots, p_{n_{\bar{M}}})^\top \in \mathbb{R}^{n_{\bar{M}}}$ to encode the values of the independent $\star$-entries of $\bar{M}$ in a particular numerical realization $\tilde{M}$.

A set $V \subseteq \mathbb{R}^n$ is called a *variety* if there exist polynomials $\varphi_1, \ldots, \varphi_k$, such that $V = \{x \in \mathbb{R}^n : \varphi_i(x) = 0, \forall i \in [k]\}$, and V is *proper* when $V \neq \mathbb{R}^n$. We denote by $V^c = \mathbb{R}^n \setminus V$ its complement.

The *term rank* [5] of a (symmetrically) structured matrix $\bar{M}$, denoted as t–rank($\bar{M}$), is the largest integer k such that, for some suitably chosen distinct rows $\{i_\ell\}_{\ell=1}^k$ and distinct columns $\{j_\ell\}_{\ell=1}^k$, all of the entries $\{[\bar{M}]_{i_\ell j_\ell}\}_{\ell=1}^k$ are $\star$-entries. Additionally, a (symmetrically) structured matrix $\bar{M} \in \{0, \star\}^{n\times m}$ is said to have *generic rank* k, denoted as g–rank($\bar{M}$) $= k$, if there exists a numerical realization $\tilde{M}$ of $\bar{M}$, such that rank($\tilde{M}$) $= k$. It is worth noting that, if g–rank($\bar{M}$) > 0, then the set of parameters describing all possible realizations form a proper variety when rank($\tilde{M}$) $<$ g–rank($\bar{M}$), [10]. We remark here that the term rank of a symmetrically structured matrix $\bar{M}$ is an upper bound of the generic rank of $\bar{M}$. Additionally, the term rank ignores the dependency among entries of $\bar{M}$ whereas the generic rank considers them.

B. Graph Theory

Given a digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, a *path* $\mathcal{P}$ in $\mathcal{D}$ is an ordered sequence of distinct vertices $\mathcal{P} = (v_1, \ldots, v_k)$ with $\{v_1, \ldots, v_k\} \subseteq \mathcal{V}$ and $(v_i, v_{i+1}) \in \mathcal{E}$ for all $i = 1, \ldots, k-1$. A *cycle* is either a path $(v_1, \ldots, v_k)$ with the additional edge (v_k, v_1) (denoted as $\mathcal{C} = (v_1, \ldots, v_k, v_1)$), or a vertex with an edge to itself (i.e., self-loop, denoted as $\mathcal{C} = (v_1, v_1)$). We denote by $\mathcal{V}_\mathcal{C} \subseteq \mathcal{V}$ the set of vertices in $\mathcal{C}$, and $\mathcal{E}_\mathcal{C} \subseteq \mathcal{E}$ the set of directed edges constituting the cycle $\mathcal{C}$. The length of a cycle $\mathcal{C}$, is defined as the number of distinct vertices in $\mathcal{C}$, i.e., the cardinality of $\mathcal{V}_\mathcal{C}$, denoted by $|\mathcal{V}_\mathcal{C}|$. Given a set $\mathcal{S}$ of vertices in $\mathcal{D}$, we let $\mathcal{D}_\mathcal{S} = (\mathcal{S}, \mathcal{S}\times\mathcal{S} \subset \mathcal{E})$ be the *subgraph of $\mathcal{D}$ induced by* $\mathcal{S}$ [26]. We say that $\mathcal{D}_\mathcal{S}$ can be covered by *disjoint cycles*, if there exists $\{\mathcal{C}_i\}_{i=1}^l$ such that $\mathcal{S} = \bigcup_{i=1}^l \mathcal{V}_{\mathcal{C}_i}$ and $\mathcal{V}_{\mathcal{C}_i} \cap \mathcal{V}_{\mathcal{C}_j} = \emptyset$, for all $i \neq j$, $i, j \in [l]$. Given a set $\mathcal{S} \subseteq \mathcal{V}$, we define the *in-neighbor set* of $\mathcal{S}$ as $\mathcal{N}(\mathcal{S}) = \{v_i \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}, v_j \in \mathcal{S}\}$.

Given a directed graph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ and two sets $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{V}$, we define the associated *bipartite graph* of $\mathcal{D}$ by $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$, whose vertex set is $\mathcal{S}_1 \cup \mathcal{S}_2$ and edge set $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2} = \{(s_1, s_2) \in \mathcal{E} : s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2\}$. Given $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$, and a set $\mathcal{S} \subseteq \mathcal{S}_1$ (or $\mathcal{S} \subseteq \mathcal{S}_2$), we define the *bipartite neighbor set* of $\mathcal{S}$ as $\mathcal{N}_\mathcal{B}(\mathcal{S}) = \{j : (j, i) \in \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}, i \in \mathcal{S}\}$. A *matching* $\mathcal{M}$ is a set of edges in $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}$ that do not share vertices, i.e., given edges $e = (s_1, s_2)$ and $e' = (s_1', s_2')$, $e, e' \in \mathcal{M}$ only if $s_1 \neq s_1'$ and $s_2 \neq s_2'$. A matching is said to be *maximum* if it is matching with the maximum number of edges among all possible matchings. Given a matching $\mathcal{M}$, two vertices s_1 and s_2 are *matched* if $e = (s_1, s_2) \in \mathcal{M}$. The vertex v is said to be *right-unmatched* (respectively, left-unmatched) with respect to a matching $\mathcal{M}$ associated with $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$ if $v \in \mathcal{S}_2$ (respectively, $v \in \mathcal{S}_1$) and v does not belong to an edge in the matching $\mathcal{M}$. We say a matching $\mathcal{M}$ is a *perfect matching* if there is no right-unmatched vertex.

IV. Graph-Theoretic Conditions for Structural Output Controllability

In this section, we provide graph-theoretical conditions for structural output controllability. Instead of taking graph-theoretic approaches as in [17] and [22], we study the problem from an algebraic geometry perspective and then interpret those conditions from a graph-theoretic perspective.

To achieve this goal, we first investigate the modes of numerical realizations of a structural pair involving symmetrically structured matrices and establish a connection with its controllability. Based on this connection, we propose graph-theoretic conditions for structural target controllability and structural output controllability in Theorems 1 and 2, respectively. Finally, in Theorem 3, we establish the NP-hardness of verifying conditions for structural output controllability.

A. Generic Properties of Symmetrically Structural Pairs

According to the definition of structural output controllability, a structured system $(\bar{A}, \bar{B}, \bar{C})$ is structurally output controllable if the output controllability matrix $Q(\tilde{A}, \tilde{B}, \tilde{C})$ is generically full rank. Since we assume that any numerical realization $\tilde{A}$ is a symmetric matrix, we have $\tilde{A}$ is diagonalizable. Hence, in the view of PBH test [25], showing that the output controllability matrix is (generically) full rank and is equivalent to showing that generically the subspace $S = \{v^\top \tilde{C} \in \mathbb{R}^n : v \in \mathbb{R}^k\}$ is spanned by the eigenvectors associated with the controllable eigenvalues of $\tilde{A}$. In this section, we first provide a characterization of the *zero* modes of numerical realizations of a structural pair in Lemma 1. Following this, we provide an algebraic condition for *nonzero* modes being generically controllable in Lemma 2.

As a first step toward characterizing the role of zero modes, we notice that the concatenation of matrices $[\tilde{A}, \tilde{B}]$ is *degenerate* if and only if there exists a nonzero vector v such that $v^\top[\tilde{A}, \tilde{B}] = 0$. In other words, given a target set $\mathcal{T}$, if $C_\mathcal{T}[\bar{A}, \bar{B}]$ is generically full rank, then any numerical realization has (almost surely) no vector v with $v^\top C_\mathcal{T}[\tilde{A}, \tilde{B}] = 0$. In Lemma 1, we first characterize the generic rank of $[\bar{A}, \bar{B}]$, which lays the foundation for the further characterization of spectral properties on nonzero modes of numerical realizations.

Lemma 1: Consider a structural pair $(\bar{A}, \bar{B})$, where $\bar{A}$ is symmetrically structured, and a target set $\mathcal{T} = \{i_1, \ldots, i_k\} \subseteq [n]$. Let $\mathcal{D}(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}(\bar{A}) \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$ be the digraph representation of $(\bar{A}, \bar{B})$, and $\mathcal{X}_\mathcal{T} \subseteq \mathcal{X}$ be the set of vertices indexed by $\mathcal{T}$. If $|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}|, \forall \mathcal{S} \subseteq \mathcal{X}_\mathcal{T}$, then *g–rank*$(C_\mathcal{T}[\bar{A}, \bar{B}]) = k$.

If we let $\mathcal{T} = [n]$, then Lemma 1 provides a sufficient condition under which g–rank($[\bar{A}, \bar{B}]$) $= n$. If $[\bar{A}, \bar{B}]$ has the full generic-rank, then there does not exist a nontrivial vector $v \in \mathbb{R}^n$ such that $v^\top[\tilde{A}, \tilde{B}] = \mathbf{0}$ for almost all numerical realization of $(\bar{A}, \bar{B})$, i.e., generically all the zero modes of a numerical realization $(\tilde{A}, \tilde{B})$ are controllable. Since we aim to derive necessary and sufficient conditions on structural controllability, what remains to be shown is when the nonzero modes of a numerical realization $(\tilde{A}, \tilde{B})$ are generically controllable.

As proved in [9], when a structural pair (with no parameter constraints) is irreducible, all the nonzero modes of almost all its numerical realizations are simple and controllable. In Lemma 2, we investigate the relationship between irreducibility and nonzero modes for structured pairs with symmetrical parameter constraints.

Lemma 2: Given a structural pair $(\bar{A}, \bar{B})$, where $\bar{A}$ is symmetrically structured, and t–rank($\bar{A}$) $= k$, if $(\bar{A}, \bar{B})$ is irreducible, then there exists a proper variety $V \subset \mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}}$, such that for any numerical realization $(\tilde{A}, \tilde{B})$ with $[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in V^c$, $\tilde{A}$ has k nonzero, simple, and controllable modes.

Remark 1: The challenge in the proof of Lemma 2 is to construct a finite number of nonzero polynomials, i.e., the polynomials of where not every coefficient is zero, such that the numerical values assigned to free parameters of $\bar{A}$ in a numerical realization $\tilde{A}$, where $\tilde{A}$ does not have k nonzero simple uncontrollable eigenvalues, are the zeros of those polynomials. Since the set of zeros of a nonzero polynomial has Lebesgue measure zero [27], it follows that for any numerical realization $\tilde{A}$, $\tilde{A}$ has almost surely k nonzero simple uncontrollable eigenvalues.

B. Solution to Problem 1

So far we have a graph-theoretic condition ensuring that all the zero modes of a numerical realization $(\tilde{A}, \tilde{B})$ are controllable generically, and an algebraic condition on irreducibility leads to controllability of nonzero simple modes almost surely. When these two conditions hold, the controllability matrix $Q(\tilde{A}, \tilde{B})$ is full rank generically. Next, we extend this idea to provide conditions for the nondegeneracy of the output controllability matrix.

By leveraging Lemmas 1 and 2, we will establish conditions under which the subspace $S = \{v^\top \tilde{C} \in \mathbb{R}^n : v \in \mathbb{R}^k\}$ is generically spanned by the eigenvectors of controllable eigenvalues of $\tilde{A}$, which implies that for any nonzero $v \in \mathbb{R}^k$, we have $v^\top \cdot Q(\tilde{A}, \tilde{B}, \tilde{C}) \neq 0$ generically. We first formalize the previous reasoning as a graph-theoretic necessary and sufficient condition for structural target controllability:

Theorem 1: Consider a structural pair $(\bar{A}, \bar{B})$, where $\bar{A}$ is symmetrically structured, and a target set $\mathcal{T} \subseteq [n]$. Let $\mathcal{X}_\mathcal{T}$ be the set of state vertices indexed by $\mathcal{T}$ in $\mathcal{D}(\bar{A}, \bar{B})$. The structural pair $(\bar{A}, \bar{B})$ is structurally target controllable with respect to $\mathcal{T}$, if and only if, the following conditions hold simultaneously in $\mathcal{D}(\bar{A}, \bar{B})$.

1) All the states vertices in $\mathcal{X}_\mathcal{T}$ are input-reachable[1].
2) There is no right-unmatched vertex in $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$ associated with $\mathcal{D}(\bar{A}, \bar{B})$, where $\mathcal{S}_1 = \mathcal{X} \cup \mathcal{U}$ and $\mathcal{S}_2 = \mathcal{X}_\mathcal{T}$.

Remark 2: Condition 2) in Theorem 1 can be verified using local information in a graph. Moreover, such a matching condition can be verified in a polynomial time $\mathcal{O}(\sqrt{|\mathcal{S}_1 \cup \mathcal{S}_2|}|\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}|)$ [28, Sec. 23.6].

By letting $\mathcal{T} = [n]$, Theorem 1 recovers the following graph-theoretic necessary and sufficient condition for structural controllability [15].

Corollary 1 (see [15]): The structural pair $(\bar{A}, \bar{B})$, where $\bar{A}$ is symmetrically structured, is structurally controllable, if and only if, the following conditions hold simultaneously in $\mathcal{D}(\bar{A}, \bar{B})$.

1) All the state vertices are input-reachable.
2) There is no right-unmatched vertex in $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$ associated with $\mathcal{D}(\bar{A}, \bar{B})$, where $\mathcal{S}_1 = \mathcal{X} \cup \mathcal{U}$ and $\mathcal{S}_2 = \mathcal{X}$.

Our characterization of structural target controllability relies on the assumption that the state matrix A is symmetric. More specifically, since the state matrix is symmetric, its eigenvectors form a complete basis of the state space, which allows us to generalize the PBH test in the context of target controllability. Such generalization cannot be applied when the state matrix is nondiagonalizable; hence, Theorem 1 is generally not true when Assumption 1 is violated (see [22, Example 3] for reference).

Notice that structural target controllability is a special case of structural output controllability [18]. More specifically, in the context of output controllability, each output is a weighted linear combination of states. To derive necessary and sufficient conditions for structural output controllability, we leverage Theorem 1, as shown in the following theorem.

Theorem 2: Consider a structural system $(\bar{A}, \bar{B}, \bar{C})$, where $\bar{A}$ is a symmetrically structured matrix, whereas $\bar{B}, \bar{C}$ are structured matrices. The structural system $(\bar{A}, \bar{B}, \bar{C})$ is structurally output controllable, if and only if, the following conditions hold simultaneously.

1) There exists a target set $\mathcal{T} \subseteq [n]$ such that $(\bar{A}, \bar{B})$ is structurally target controllable with respect to $\mathcal{T}$.
2) There is no right-unmatched vertex in $\mathcal{B}(\mathcal{X}_\mathcal{T}, \mathcal{Y}, \mathcal{E}_{\mathcal{X}_\mathcal{T},\mathcal{Y}})$, where $\mathcal{Y} = \{y_i\}_{i=1}^k$, $\mathcal{X}_\mathcal{T} = \{x_i \in \mathcal{X} : i \in \mathcal{T}\}$, and $\mathcal{E}_{\mathcal{X}_\mathcal{T},\mathcal{Y}} = \{\{x_j, y_i\} : [\bar{C}]_{ij} = \star\}$.

The conditions in Theorem 2 require us to find a target set $\mathcal{T}$ for which a matching condition in a bipartite graph $\mathcal{B}(\mathcal{X}_\mathcal{T}, \mathcal{Y}, \mathcal{E}_{\mathcal{X}_\mathcal{T},\mathcal{Y}})$ is satisfied. Naively, there are exponentially many possible target sets $\mathcal{T}$, implying that it may be computationally challenging to verify structural output controllability through the conditions in Theorem 2. Indeed, we show in Theorem 3 that verifying those conditions is NP-hard.

[1]We say a state vertex $x_i \in \mathcal{X}$ is (input)-reachable if there exists a path from an input vertex $u_j \in \mathcal{U}$ to x_i.

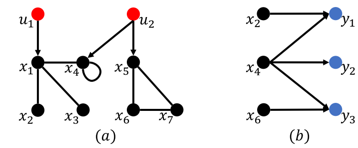

Fig. 1. (a) Mixed graph representation of the structural pair $(\bar{A}, \bar{B})$, where the red and black vertices represent input and state vertices, respectively. The black lines and arrows represent edges in $\mathcal{G}(\bar{A}, \bar{B})$. (b) Bipartite graph $\mathcal{B}(\mathcal{X}_\mathcal{T}, \mathcal{Y}, \mathcal{E}_{\mathcal{X}_\mathcal{T},\mathcal{Y}})$, where $\mathcal{X}_\mathcal{T} = \{x_2, x_4, x_6\}$ and $\mathcal{Y} = \{y_1, y_2, y_3\}$. The black and blue vertices are target vertices $\mathcal{X}_\mathcal{T}$ and output vertices $\mathcal{Y}$, respectively.

Theorem 3: Consider a structural system $(\bar{A}, \bar{B}, \bar{C})$, where $\bar{A} \in \{0, \star\}^{n \times n}$ is a symmetrically structured matrix. The problem of verifying the necessary and sufficient conditions in Theorem 2 is NP-hard.

V. Illustrative Examples

In this section, we consider a few examples to illustrate Theorems 1 and 2.

We consider a symmetrically structured system with 7 states, 2 inputs, and 3 outputs. Let the target set be $\mathcal{T} = \{2, 4, 6\}$. The structural representations of the state, input, output, and target matrices are as follows:

$$\bar{A} = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 \\ a_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{14} & 0 & 0 & a_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{56} & a_{57} \\ 0 & 0 & 0 & 0 & a_{56} & 0 & a_{67} \\ 0 & 0 & 0 & 0 & a_{57} & a_{67} & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} b_{11} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & b_{42} \\ 0 & b_{52} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 0 & c_{12} & 0 & c_{14} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{34} & 0 & c_{36} & c_{37} \end{bmatrix}, \text{and } C_\mathcal{T} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We also associate the structural pair $(\bar{A}, \bar{B})$ with the mixed graph $\mathcal{G}(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_u(\bar{A}), \mathcal{E}_{\mathcal{U},\mathcal{X}})$, depicted in Fig. 1, where $\mathcal{X} = \{x_i\}_{i=1}^7$, $\mathcal{U} = \{u_1, u_2\}$ and $\mathcal{X}_\mathcal{T} = \{x_2, x_4, x_6\}$. Since all the vertices in $\mathcal{X}_\mathcal{T}$ are input-reachable, and $|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}|, \ \forall \mathcal{S} \subseteq \mathcal{X}_\mathcal{T}$, by Theorem 1, $(\bar{A}, \bar{B})$ is structurally target controllable with respect to $\mathcal{T}$. This example also shows that if the input-reachability of the vertices in $\mathcal{X}_\mathcal{T}$ is guaranteed, then the structural target controllability in undirected networks can be verified by only local topological information. Finally, to verify the structural output controllability, we notice that there exists a target set $\mathcal{T} = \{2, 4, 6\}$ such that $(\bar{A}, \bar{B})$ is structurally target controllable with respect to $\mathcal{T}$ and there is no right-unmatched vertex with respect to any maximum matching in $\mathcal{B}(\mathcal{X}_\mathcal{T}, \mathcal{Y}, \mathcal{E}_{\mathcal{X}_\mathcal{T},\mathcal{Y}})$, where $\mathcal{Y} = \{y_1, y_2, y_3\}$ is the set of output vertices. By Theorem 2, $(\bar{A}, \bar{B}, \bar{C})$ is structurally output controllable.

VI. Conclusion

In this article, we study the problem of characterizing structural output controllability in structured systems with symmetric state matrices, such as undirected networks. To address this problem, we first characterized the generic properties of symmetrically structured matrices. Based on this, we derived necessary and sufficient conditions for structural target controllability and structural output controllability of undirected networks. Although verifying the proposed conditions on structural target controllability is in polynomial time, we showed that verifying the proposed conditions on structural output controllability is NP-hard.

Appendix

A. Proof of Lemma 1

Before we proceed to the proof of Lemma 1, we introduce Proposition 1 and Lemma 3, which lay the foundation for the proof of Lemma 1.

Proposition 1 ([5, Sec. 1.2]): Given a (symmetrically) structured matrix $\bar{M} \in \{0, \star\}^{n\times m}$ and $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$, where $\mathcal{S}_1 = \{v_1, \dots, v_m\}$, $\mathcal{S}_2 = \{v'_1, \dots, v'_n\}$, and $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2} = \{\{v_i, v'_j\} : [\bar{M}]_{ji} \neq 0, v_i \in \mathcal{S}_1, v'_j \in \mathcal{S}_2\}$, then *t–rank*$(\bar{M}) = n$ if and only if $|\mathcal{N}_{\mathcal{B}}(\mathcal{S})| \geq |\mathcal{S}|$ for all $\mathcal{S} \subseteq \mathcal{S}_2$.

Lemma 3: Consider an $n \times n$ symmetrically structured matrix $\bar{A}$, and a set $\mathcal{T} = \{i_1, \dots, i_k\} \subseteq [n]$. Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ be the digraph representation of $\bar{A}$, $\mathcal{X}_{\mathcal{T}} \subseteq \mathcal{X}$ be the set of vertices indexed by $\mathcal{T}$, and $C_{\mathcal{T}}$ be defined as in (2). The generic-rank of $C_{\mathcal{T}}\bar{A}$ equals k if and only if $|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}|, \forall \mathcal{S} \subseteq \mathcal{X}_{\mathcal{T}}$.

Proof of Lemma 3: First, we show the sufficiency of the theorem. Notice that the generic-rank of $C_{\mathcal{T}}\bar{A}$ equals k, if and only if, there exists a k-by-k nonzero minor in $C_{\mathcal{T}}\bar{A}$; hence, it suffices to find that minor. Since $|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}|, \forall \mathcal{S} \subseteq \mathcal{X}_{\mathcal{T}}$, there exist k entries that lie on distinct rows and distinct columns of $C_{\mathcal{T}}\bar{A}$ according to Proposition 1. As a result, we can select rows indexed by $\mathcal{T} = \{i_1, \dots, i_k\}$ and columns indexed $j_1, \dots, j_k$ in $\bar{A}$ such that $\{[\bar{A}]_{i_\ell j_\ell}\}_{\ell=1}^k$ lies on distinct rows and distinct columns. Next, we consider the following two cases.

On one hand, if $\{j_1, \dots, j_k\} = \{i_1, \dots, i_k\}$, then $M = C_{\mathcal{T}}\bar{A}C_{\mathcal{T}}^\top$ is a square submatrix of $\bar{A}$. We consider a particular numerical realization $\tilde{A}$ of $\bar{A}$, as follows. Let $[\tilde{A}]_{ij} \neq 0$ for all $(i,j) \notin \{(i_\ell, j_\ell) : \ell \in [k]\}$, $[\tilde{A}]_{ij} = [\tilde{A}]_{ji}$, and $[\tilde{A}]_{ij} = 0$ otherwise. Subsequently, by computing the determinant, $\det(C_{\mathcal{T}}\tilde{A}C_{\mathcal{T}}^\top) = \text{sgn}(\sigma_1)\Pi_{\ell=1}^k[\tilde{A}]_{i_\ell j_\ell} + \text{sgn}(\sigma_2)\Pi_{\ell=1}^k[\tilde{A}]_{j_\ell i_\ell}$, where $\text{sgn}(\sigma_1)$ and $\text{sgn}(\sigma_2)$ are the signatures of the permutations $\sigma_1 = \{(i_\ell, j_\ell) : \ell \in [k]\}$, and $\sigma_2 = \{(j_\ell, i_\ell) : \ell \in [k]\}$, respectively. Notice that if $\text{sgn}(\sigma_1) = \text{sgn}(\sigma_2)$, then it follows that $\det(C_{\mathcal{T}}\tilde{A}C_{\mathcal{T}}^\top) \neq 0$. Furthermore, if $\{\tilde{A} : \det(C_{\mathcal{T}}\tilde{A}C_{\mathcal{T}}^\top) = 0\}$ is a proper variety, we have that M admits an k-by-k nonzero minor generically. Thus, the generic-rank of $C_{\mathcal{T}}\bar{A}$ equals k.

On the other hand, when $\{j_1, \dots, j_k\} \neq \{i_1, \cdots, i_k\}$, it suffices to show there exists a numerical realization $\tilde{A}$ such that $\det([\tilde{A}]_{i_1,\dots,i_k}^{j_1,\dots,j_k}) \neq 0$. We consider a numerical realization $\tilde{A}$ by assigning distinct real values to $\star$-entries corresponding to $\{[\bar{A}]_{i_\ell j_\ell}\}_{\ell=1}^k$ while keeping $[\tilde{A}]_{ij} = [\tilde{A}]_{ji}$, and assigning 0 otherwise. Without loss of generality, we can permute $\{\ell\}_{\ell=1}^k$ such that for each $[\bar{A}]_{i_{\ell_r} j_{\ell_r}} \in \{[\bar{A}]_{i_{\ell_r} j_{\ell_r}}\}_{r=1}^p$, $[\bar{A}]_{j_{\ell_r} i_{\ell_r}}$ is not in matrix $[\bar{A}]_{i_1,\dots,i_k}^{j_1,\dots,j_k}$, and for each $[\bar{A}]_{i_{\ell_r} j_{\ell_r}} \in \{[\bar{A}]_{i_{\ell_r} j_{\ell_r}}\}_{r=p+1}^k$, $[\bar{A}]_{j_{\ell_r} i_{\ell_r}}$ is in matrix $[\bar{A}]_{i_1,\dots,i_k}^{j_1,\dots,j_k}$. We declaim that there is only one nonzero entry in either the i_{ℓ_r}th row or j_{ℓ_r}th column, $\forall r \in [p]$, otherwise it contradicts that $\{[\bar{A}]_{i_\ell j_\ell}\}_{\ell=1}^k$ are in distinct rows and distinct columns of $[\bar{A}]$. Thus, we compute $\det([\tilde{A}]_{i_1,\dots,i_k}^{j_1,\dots,j_k})$

$$\det\left([\tilde{A}]_{i_1,\dots,i_k}^{j_1,\dots,j_k}\right) = \left(\prod_{r=1}^{p}[\tilde{A}]_{i_{\ell_r} j_{\ell_r}}\right) \cdot \det\left([\tilde{A}]_{i_{\ell_{p+1}},\dots,i_{\ell_k}}^{j_{\ell_{p+1}},\dots,j_{\ell_k}}\right) \neq 0 \tag{4}$$

where $\det([\tilde{A}]_{i_{\ell_{p+1}},\dots,i_{\ell_k}}^{j_{\ell_{p+1}},\dots,j_{\ell_k}}) \neq 0$ is true because of the reasoning in the first case $\{i_1, \dots, i_k\} = \{j_1, \dots, j_k\}$. Thus, there exists numerical realization such that $\det([\tilde{A}]_{i_1,\dots,i_k}^{j_1,\dots,j_k}) \neq 0$.

Next, we show the necessity of the theorem by contrapositive. We assume that there exists $\mathcal{S} \subseteq \mathcal{X}_{\mathcal{T}}$, such that $|\mathcal{N}(\mathcal{S})| < |\mathcal{S}|$. Then, by Proposition 1, there does not exist k entries that lie on the distinct rows and distinct columns of $C_{\mathcal{T}}\bar{A}$, which implies g–rank$(C_{\mathcal{T}}\bar{A}) < k$. ■

Proof of Lemma 1: Suppose $|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}|$, $\forall \mathcal{S} \subseteq \mathcal{X}_{\mathcal{T}}$, then, by Proposition 1, there exist k entries, $\{[\bar{A}, \bar{B}]_{i_\ell j_\ell}\}_{\ell=1}^k$, such that they are all $\star$-entries that lie on distinct rows and distinct columns of $[\bar{A}, \bar{B}]$. Among those k entries, suppose $\{[\bar{A}, \bar{B}]_{i_\ell j_\ell}\}_{\ell=1}^q$ are in columns of $\bar{A}$, and $\{[\bar{A}, \bar{B}]_{i_\ell j_\ell}\}_{\ell=q+1}^k$ are in the columns of $\bar{B}$. By Lemma 3, there exists a numerical realization $\tilde{A}$, such that $\det([\tilde{A}, \tilde{B}]_{i_1,\dots,i_q}^{j_1,\dots,j_q}) \neq 0$. Since $\bar{B}$ is a structured matrix, there exists a numerical realization $\tilde{B}$ such that $\det([\tilde{A}, \tilde{B}]_{i_{q+1},\dots,i_k}^{j_{q+1},\dots,j_k}) \neq 0$ Hence, there exists a numerical realization $[\tilde{A}, \tilde{B}]$ with

$$\det([\tilde{A}, \tilde{B}]_{i_1,\dots,i_k}^{j_1,\dots,j_k}) = \det([\tilde{A}, \tilde{B}]_{i_1,\dots,i_q}^{j_1,\dots,j_q})\det([\tilde{A}, \tilde{B}]_{i_{q+1},\dots,i_k}^{j_{q+1},\dots,j_k}) \neq 0$$

which implies that g–rank$(C_{\mathcal{T}}[\bar{A}, \bar{B}]) = k$.

B. Proof of Lemma 2

We introduce Proposition 2, Proposition 3, Lemma 4, Lemma 5, and Lemma 6 to support the proof of Lemma 2.

Proposition 2 ([29, Sec. 2.1]): Let $\varphi_1(s)$ and $\varphi_2(s)$ be polynomials in s with $\varphi_1(s) = \sum_{i=0}^{n_1} a_i s^{n_1-i}$, and $\varphi_2(s) = \sum_{i=0}^{n_2} b_i s^{n_2-i}$, respectively. Let $R(\varphi_1, \varphi_2)$ be defined as

$$R(\varphi_1, \varphi_2) = \det\left(\left[\begin{array}{ccccccc} a_{n_1} & a_{n_1-1} & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & a_{n_1} & \cdots & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n_1} & a_{n_1-1} & \cdots & a_0 \\ \hline 0 & 0 & \cdots & & & \cdots & b_0 \\ \vdots & \vdots & \iddots & \vdots & \vdots & \iddots & \vdots \\ 0 & b_{n_2} & \cdots & b_1 & b_0 & \cdots & 0 \\ b_{n_2} & b_{n_2-1} & \cdots & b_0 & 0 & \cdots & 0 \end{array}\right]\right). \tag{5}$$

If $a_{n_1} \neq 0$ and $b_{n_2} \neq 0$, then $\varphi_1(s)$ and $\varphi_2(s)$ have a nontrivial common factor if and only if the $R(\varphi_1, \varphi_2) = 0$.

Proposition 3 (Hoffman–Wielandt Theorem [30, Sec. 6.3]): Given $n \times n$ symmetric matrices A and E, let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A, and $\hat{\lambda}_1, \dots, \hat{\lambda}_n$ be the eigenvalues of $A + E$. There is a permutation $\sigma(\cdot)$ of the integers $\{1, \dots, n\}$ such that

$$\sum_{i=1}^{n}(\hat{\lambda}_{\sigma(i)} - \lambda_i)^2 \leq \|E\|_F^2 \tag{6}$$

where $\|E\|_F = \sqrt{\text{tr}(EE^\top)}$.

Lemma 4: Let $\bar{A}$ be an $n \times n$ symmetrically structured matrix, and let $\mathcal{D}(\bar{A}) = \{\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}}\}$ be the digraph associated with $\bar{A}$. Assume *t–rank*$(\bar{A}) = k$, and denote $\{[\bar{A}]_{i_\ell j_\ell}\}_{\ell=1}^k$ as the k entries that lie on distinct rows and distinct columns. We define $\mathcal{S} = \{x_{i_1}, \dots, x_{i_k}\} \subseteq \mathcal{X}$. Then, $\mathcal{D}_{\mathcal{S}}$ can be covered by disjoint cycles.

Proof of Lemma 4: We approach the proof by contradiction. Suppose $\mathcal{D}_{\mathcal{S}}$ cannot be covered by disjoint cycles, then at least one vertex $x_i \in \mathcal{S}$ can only be covered by cycles intersecting with other cycles in $\mathcal{D}_{\mathcal{S}}$, which implies that there does not exist k edges in which no two edges share the same "tail" or "head" vertex in $\mathcal{D}(\bar{A})$, i.e., there does not exist k entries that lie on distinct rows and distinct columns of $\bar{A}$, which, by Proposition 1, contradicts t–rank$(\bar{A}) = k$. ■

Lemma 5: Given an $n \times n$ symmetrically structured matrix $\bar{A}$, if *t–rank*$(\bar{A}) = k$, then there exists a proper variety $V_1 \subset \mathbb{R}^{n_{\bar{A}}}$, such that for any numerical realization $\tilde{A}$, where the numerical values assigned to free parameters of $\bar{A}$ are encoded in the vector $\mathbf{p}_{\tilde{A}} \in \mathbb{R}^{n_{\bar{A}}} \setminus V_1$, $\tilde{A}$ has k nonzero simple eigenvalues.

Proof of Lemma 5: We expand the characteristic polynomial of a matrix $\tilde{A}$ as

$$\det(sI - \tilde{A}) = s^n + a_{n-1}s^{n-1} \cdots + a_{n-k}s^{n-k} + \cdots + a_0. \tag{7}$$

Besides, we have

$$a_q = (-1)^{n-q} \sum_{1\le k_1<\ldots<k_{n-q}\le n} \det([\tilde{A}]^{k_1,\ldots,k_{n-q}}_{k_1,\ldots,k_{n-q}}) \quad (8)$$

where $q = 0, 1, \ldots, n-1$. Since t–rank$(\bar{A}) = k$, there exists a numerical realization $\tilde{A}$ and a set of indexes, $\{i_1, \ldots, i_k\} \subseteq [n]$, such that $\det([\tilde{A}]^{i_1,\ldots,i_k}_{i_1,\ldots,i_k}) \neq 0$. Furthermore, $V_0 := \{\mathbf{p}_{\tilde{A}} \in \mathbb{R}^{n_{\bar{A}}} : a_{n-k} = 0\}$ is a proper variety. Since the maximum order of principle minor is at most the term rank of a matrix, we have $a_{n-k-1} = \cdots = a_0 = 0$. Thus, to characterize nonzero eigenvalues, we define the polynomial $\varphi_{\tilde{A}}(s)$ as

$$\varphi_{\tilde{A}}(s) = s^k + a_{n-1}s^{k-1} + \cdots + a_{n-k}. \quad (9)$$

In the rest of the proof, we show that there exists a numerical realization $\mathbf{p}_{\tilde{A}} \in V_0^c$ such that $\tilde{A}$ has k nonzero simple eigenvalues. Since t–rank$(\bar{A}) = k$, we define the set $\mathcal{S}$ as in Lemma 4. By Lemma 4, there exist disjoint cycles $\mathcal{C}_1, \ldots, \mathcal{C}_l$ covering $\mathcal{D}_{\mathcal{S}}$. Let us denote by $\mathcal{C}_i$ the ith cycle in $\{\mathcal{C}_1, \ldots, \mathcal{C}_l\}$. Moreover, without loss of generality, we let the length of cycle $\mathcal{C}_i$ be either $|\mathcal{C}_i| = 2q$, or $|\mathcal{C}_i| = 2q+1$, for some $q \in \mathbb{N}$. Note that by definition, there is a one-to-one correspondence between the edge in $\mathcal{D}(\bar{A})$ and the $\star$-entry in $\bar{A}$. From this observation, we denote by $\bar{A}_i \in \{0, \star\}^{|\mathcal{C}_i|\times|\mathcal{C}_i|}$ the square submatrix formed by collecting rows and columns corresponding to the indexes of vertices in $\mathcal{V}_{\mathcal{C}_i}$ of the cycle $\mathcal{C}_i$. We let all the $\star$-entries of $\bar{A}$ be zero, except for $\star$-entries corresponding to edges in $\{\mathcal{E}_{\mathcal{C}_i}\}_{i=1}^l$. Hence, there exists a permutation matrix P and numerical realization $\tilde{A}$, such that $P\tilde{A}P^{-1}$ is a block diagonal matrix

$$P\tilde{A}P^{-1} = \begin{bmatrix} \tilde{A}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{A}_2 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \tilde{A}_l & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix}. \quad (10)$$

If $|\mathcal{C}_i| = 2q$, without loss of generality, we could assume $\mathcal{C}_i = (x_{i_1}, x_{j_1}, x_{i_2}, x_{j_2}, \ldots, x_{i_q}, x_{j_q}, x_{i_1})$. Since $\mathcal{D}_{\mathcal{V}_{\mathcal{C}_i}}$ is a subgraph of the digraph $\mathcal{D}(\bar{A})$ associated with the symmetrically structured matrix $\bar{A}$, there exist q disjoint cycles of length-2 covering $\mathcal{D}_{\mathcal{V}_{\mathcal{C}_i}}$, i.e., cycles $(x_{i_1}, x_{j_1}, x_{i_1}), (x_{i_2}, x_{j_2}, x_{i_2}), \ldots, (x_{i_q}, x_{j_q}, x_{i_q})$. We assign distinct nonzero weights to $\star$-entries of $\bar{A}_i$ that correspond to edges in the q cycles of length-2, and assign zero weights to other $\star$-entries in $\bar{A}_i$. As a result, we have

$$\tilde{A}_i = \begin{bmatrix} \boxed{\begin{matrix} 0 & a_{i_1j_1} \\ a_{i_1j_1} & 0 \end{matrix}} & \cdots & 0 & 0 \\ & \cdots & 0 & 0 \\ \vdots & \vdots\ \ddots & \vdots & \vdots \\ 0 & 0\ \cdots & \boxed{\begin{matrix} 0 & a_{i_qj_q} \\ a_{i_qj_q} & 0 \end{matrix}} & \\ 0 & 0\ \cdots & & \end{bmatrix},$$

where $a_{i_1j_1}, \ldots, a_{i_qj_q}$ are q nonzero distinct weights. Thus, $\tilde{A}_i$ has $2q$ simple nonzero eigenvalues.

If $|\mathcal{C}_i| = 1$, then the eigenvalue of $\tilde{A}_i \in \mathbb{R}^{1\times1}$ can be placed to any value. If $|\mathcal{C}_i| = 2q+1$ and $q > 0$, then there are $2q$ vertices in $\mathcal{C}_i$ that can be covered by q cycles of length-2, and one vertex that cannot be covered by any length-2 cycle in a vertex-disjoint way in $\mathcal{D}_{\mathcal{V}_{\mathcal{C}_i}}$. Assign distinct nonzero weights to $\star$-entries corresponding to the q cycles of length-2, and zero to other $\star$-entries in $\bar{A}_i$. As a result, the constructed numerical realization $\tilde{A}_i$ has $2q$ nonzero simple eigenvalues and one zero eigenvalue. Denote by $\lambda_j(\tilde{A}_i)$, the jth eigenvalue of $\tilde{A}_i$, $j \in \{1, \ldots, |\mathcal{C}_i|\}$.

By Proposition 3, given a sufficiently small $\epsilon > 0$, $\exists\delta > 0$ and permutation $\sigma(\cdot)$ of integers $\{1, \ldots, |\mathcal{C}_i|\}$, such that for two numerical realizations of $\bar{A}_i$: $\tilde{A}_i$ and $\tilde{A}_{ip}$, if $||\tilde{A}_{ip} - \tilde{A}_i||_F < \delta$, then $\max\{|\lambda_{\sigma(j)}(\tilde{A}_{ip}) - \lambda_j(\tilde{A}_i)|\} < \epsilon$. Perturb $\star$-entries of $\tilde{A}_i$ corresponding to edges in $\mathcal{E}_{\mathcal{C}_i}$, such that $\tilde{A}_{ip}$, which is derived by this perturbation of $\tilde{A}_i$, satisfies $||\tilde{A}_{ip} - \tilde{A}_i||_F < \delta$. Moreover, since t–rank$(\bar{A}_i) = 2q+1$, by Lemma 3, g–rank$(\bar{A}_i) = 2q+1$. The above analysis shows that we can perturb $\tilde{A}_i$, such that rank$(\tilde{A}_{ip}) = 2q+1$, and

$$\min_{j\neq r, j, r\in\{1,\ldots,|\mathcal{C}_i|\}} |\lambda_j(\tilde{A}_{ip}) - \lambda_r(\tilde{A}_{ip})| >$$
$$\min_{j\neq r, j, r\in\{1,\ldots,|\mathcal{C}_i|\}} |\lambda_j(\tilde{A}_i) - \lambda_r(\tilde{A}_i)| - 2\epsilon.$$

It implies that there exists $\tilde{A}_{ip}$ which has $2q+1$ nonzero simple eigenvalues. Notice that $\tilde{A}_{ip}$ is also a numerical realization of $\bar{A}_i$. Hence, for either $|\mathcal{C}_i| = 2q$, or $|\mathcal{C}_i| = 2q+1$, there exists a numerical realization $\tilde{A}_i$ such that $\tilde{A}_i$ has $|\mathcal{C}_i|$ nonzero simple eigenvalues. Also, there exists $\tilde{A}$ that has $\sum_{i=1}^l |\mathcal{C}_i| = k$ nonzero simple eigenvalues.

Denote by $\varphi'_{\tilde{A}}$ the derivative of $\varphi_{\tilde{A}}$ with respect to λ. If $\mathbf{p}_{\tilde{A}} \in V_0^c$, and $\tilde{A}$ has repeated nonzero modes, then $\varphi_{\tilde{A}}$ and $\varphi'_{\tilde{A}}$ have a common nontrivial zero (i.e., by Proposition 2, $R(\varphi_{\tilde{A}}, \varphi'_{\tilde{A}}) = 0$). Define $V_1 = \{\mathbf{p}_{\tilde{A}} \in \mathbb{R}^{n_{\bar{A}}} : a_{n-k} = 0$ or $R(\varphi_{\tilde{A}}, \varphi'_{\tilde{A}}) = 0\}$, where $a_{n-k} = 0$ and $R(\varphi_{\tilde{A}}, \varphi'_{\tilde{A}}) = 0$ are both polynomials of $\star$-entries of $\bar{A}$. Since we have shown that there exists $\tilde{A}$, which has k nonzero simple eigenvalues, i.e., $\exists\mathbf{p}_{\tilde{A}} \in \mathbb{R}^{n_{\bar{A}}}$ such that $a_{n-k} \neq 0$ and $R(\varphi_{\tilde{A}}, \varphi'_{\tilde{A}}) \neq 0$, we conclude that V_1 is proper.

Remark 3: To characterize the generic rank of $[\bar{A}, \bar{B}]$, which is crucial in the derivation of Lemma 2, we should consider the proper variety in parameter space $\mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}}$. Since each $\star$-entry of $\bar{A}$ is independent of those in $\bar{B}$, V_1 is also a proper variety in $\mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}}$. Let us redefine V_1 as

$$V_1 = \{[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in \mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}} : a_{n-k} = 0 \textit{ or } R(\varphi_{\tilde{A}}, \varphi'_{\tilde{A}}) = 0\}. \quad (11)$$

Lemma 6: Consider an irreducible structural pair $(\bar{A}, \bar{B})$, where $\bar{A} \in \{0, \star\}^{n\times n}$ is a symmetrically structured matrix with *t–rank*$(\bar{A}) = k$. Let $V_1 \subset \mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}}$ be defined as in (11). There exists a proper variety $V_2 \subset \mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}}$ such that if $[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in V_1^c$, then there exists a nonzero uncontrollable mode of $\tilde{A}$ if and only if $[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in V_2$.

Sketch of Proof of Lemma 6: We will first prove that V_2 exists. Suppose $[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in V_1^c$, by a similar reasoning as in Lemma 5, all the k nonzero eigenvalues of $\tilde{A}$ are simple. Let λ be a nonzero eigenvalue of $\tilde{A}$, and $\varphi_{\tilde{A}}(s)$ be defined as in (9), then we have

$$\varphi_{\tilde{A}}(\lambda) = \lambda^k + a_{n-1}\lambda^{k-1} + \cdots + a_{n-k} = 0. \quad (12)$$

Let us further assume that (λ, v) is an uncontrollable mode of $\tilde{A}$; in other words

$$v^\top\tilde{A} = \lambda v^\top, \ \ v^\top\tilde{B} = \mathbf{0}. \quad (13)$$

Since all the nonzero eigenvalues λ are simple, recall the fact in [9] that the left eigenvector $v^\top$ equals (apart from a constant scalar) any of the nonzero row of the adjugate matrix adj$(\lambda I - \tilde{A})$. Hence

$$\text{adj}(\lambda I - \tilde{A})\tilde{B} = \mathbf{0}_{n\times m}. \quad (14)$$

Equations (12) and (14) imply that the two polynomials (15) and (16) have a common zero λ, namely

$$\varphi_{\tilde{A}}(s) = s^k + a_{n-1}s^{k-1} + \cdots + a_{n-k} = 0 \quad (15)$$

$$\psi_{\tilde{A},\tilde{B}}(s) = \text{tr}([\text{adj}(sI - \tilde{A})\tilde{B}][\text{adj}(sI - \tilde{A})\tilde{B}]^\top) = 0. \quad (16)$$

The variety V_2 is defined as follows:

$$V_2 = \{[\mathbf{p}_{\bar{A}}, \mathbf{p}_{\bar{B}}] \in \mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}} : R(\varphi_{\tilde{A}}, \psi_{\tilde{A},\tilde{B}}) = 0\} \quad (17)$$

where $R(\varphi_{\tilde{A}}, \psi_{\tilde{A},\tilde{B}}) = 0$ is a polynomial of the $\star$-entries in $\bar{A}$ and $\bar{B}$. The properness of V_2 can be shown by contradiction by adapting the proof in [9, Th. 2]. Conversely, suppose $[\mathbf{p}_{\bar{A}}, \mathbf{p}_{\bar{B}}] \in V_2 \cap V_1^c$, by the definition of V_1 and V_2, $\varphi_{\tilde{A}}$ and $\psi_{\tilde{A},\tilde{B}}$ have a common zero $\lambda \neq 0$. Since λ is a zero of $\varphi_{\tilde{A}}$, λ is also an eigenvalue of $\tilde{A}$, which is an uncontrollable eigenvalue. ■

Proof of Lemma 2: Define $V = V_1 \cup V_2$, where V_1 and V_2 are defined as in (11) and (17), respectively. We can prove V_1 is proper by a similar reasoning as the one in Lemma 5. By Lemma 6, V_2 is proper. Hence, $V = V_1 \cup V_2$ is proper. If $[\mathbf{p}_{\bar{A}}, \mathbf{p}_{\bar{B}}] \in V^c$, $\tilde{A}$ has k nonzero simple controllable modes. ■

C. Proof of Theorem 1

We first introduce Lemma 7, which lays the foundation for the proof of Theorem 1.

Lemma 7: Consider a structural pair $(\bar{A}, \bar{B})$, and a target set $\mathcal{T}$. Let $\mathcal{X}_\mathcal{T} \subseteq \mathcal{X}$ be the set of target vertices in $\mathcal{D}(\bar{A}, \bar{B})$. Let $C_\mathcal{T}$ be defined as in (2). Then, for any numerical realization $(\tilde{A}, \tilde{B})$, we have that *rank*$(C_\mathcal{T} Q(\tilde{A}, \tilde{B})) \leq |\mathcal{N}(\mathcal{X}_\mathcal{T})|$.

Proof of Lemma 7: Suppose we have a numerical realization $(\tilde{A}, \tilde{B})$. By Cayley–Hamilton theorem

$$\begin{aligned}\operatorname{rank}(C_\mathcal{T}[\tilde{B}, \tilde{A}\cdot Q(\tilde{A}, \tilde{B})]) &= \operatorname{rank}(C_\mathcal{T}[\tilde{B}, \tilde{A}\tilde{B}, \ldots, \tilde{A}^n\tilde{B}]) \\ &= \operatorname{rank}([C_\mathcal{T}\cdot Q(\tilde{A}, \tilde{B}), C_\mathcal{T}\cdot \tilde{A}^n\tilde{B}]) \\ &= \operatorname{rank}(C_\mathcal{T}\cdot Q(\tilde{A}, \tilde{B})). \end{aligned} \quad (18)$$

In $\mathcal{D}(\bar{A}, \bar{B})$, let m_1, m_2 be the number of input, state vertices in $\mathcal{N}(\mathcal{X}_\mathcal{T})$, respectively. Then, (18) yields

$$\begin{aligned}\operatorname{rank}(C_\mathcal{T}\cdot Q(\tilde{A}, \tilde{B})) &= \operatorname{rank}(C_\mathcal{T}[\tilde{B}, \tilde{A}\cdot Q(\tilde{A}, \tilde{B})]) \\ &\leq \operatorname{rank}(C_\mathcal{T}\tilde{B}) + \operatorname{rank}(C_\mathcal{T}\tilde{A}\cdot Q(\tilde{A}, \tilde{B})) \\ &\leq m_1 + \min(\operatorname{rank}(C_\mathcal{T}\tilde{A}), \operatorname{rank}(Q(\tilde{A}, \tilde{B}))) \\ &\leq m_1 + m_2 = |\mathcal{N}(\mathcal{X}_\mathcal{T})|.\end{aligned}$$

■

Proof of Theorem 1: We first show the necessity of the two conditions. Let $C_\mathcal{T}$ be defined as in (2). On one hand, suppose there exists a vertex $v_i \in \mathcal{X}_\mathcal{T}$ that is not input-reachable, then the ith row of controllability matrix will be zero row, which implies that $\operatorname{rank}(Q(\tilde{A}, \tilde{B}, C_\mathcal{T})) < |\mathcal{T}|$, for any numerical realization of the pair $(\bar{A}, \bar{B})$. On the other hand, recall that by Proposition 1, the condition 2) is satisfied if and only if $|\mathcal{N}(S)| \geq |\mathcal{S}|$, for $\forall \mathcal{S} \subseteq \mathcal{X}_\mathcal{T}$. Suppose that there exists a set $\mathcal{S} \subseteq \mathcal{X}_\mathcal{T}$, such that $|\mathcal{N}(\mathcal{S})| < |\mathcal{S}|$, then by a similar reasoning used in Lemma 7, $\operatorname{rank}(Q(\tilde{A}, \tilde{B}, C_\mathcal{T})) < |\mathcal{T}|$, for any numerical realization of the pair $(\bar{A}, \bar{B})$. Hence, the violation of Condition 1) or Condition 2) implies that generically $Q(\tilde{A}, \tilde{B}, C_\mathcal{T})$ has rank lower than $|\mathcal{T}|$, i.e., $(\bar{A}, \bar{B})$ is not structurally target controllable with respect to $\mathcal{T}$. The necessity is proved.

What remains to be shown is their sufficiency. It suffices to show that Conditions 1) and 2) result in that generically the left null space of $Q(\tilde{A}, \tilde{B}, C_\mathcal{T})$ is trivial. Suppose there exists an input-unreachable state vertex $x_i \in \mathcal{X} \setminus \mathcal{X}_\mathcal{T}$. Since all the vertices in $\mathcal{X}_\mathcal{T}$ are input-reachable, for $\forall x_j \in \mathcal{X}_\mathcal{T}$, there is no path from x_j to x_i, and there is also no path from x_i to x_j due to the symmetry in $\mathcal{D}(\bar{A})$. This implies in model (1) that the ith state has no impact on the dynamics of $\mathcal{T}$ corresponding states. Omitting the ith state from the system will not change the dynamics of $\mathcal{T}$ corresponding states. Hence, we could assume that $(\bar{A}, \bar{B})$ is irreducible. By Lemma 2, there exists a proper variety $V \subset \mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}}$, such that if $[\mathbf{p}_{\bar{A}}, \mathbf{p}_{\bar{B}}] \in V^c$, then all the nonzero modes of $\tilde{A}$ are controllable. In the rest of the proof, we assume $[\mathbf{p}_{\bar{A}}, \mathbf{p}_{\bar{B}}] \in V^c$. Denote by $e_1, \ldots, e_l$ the left eigenvectors corresponding to zero modes of $\tilde{A}$, and $e_{l+1}, \ldots, e_n$ the left eigenvectors for nonzero modes. Denote the left null space of a matrix M as $\boldsymbol{N}(M^\top)$.

From Lemma 2, we have that if $[\mathbf{p}_{\bar{A}}, \mathbf{p}_{\bar{B}}] \in V^c$, then $\boldsymbol{N}((Q(\tilde{A}, \tilde{B}))^\top) \subseteq \operatorname{span}\{e_1^\top, \ldots, e_l^\top\}$. For the target set $\mathcal{T}$, define the matrix $C_\mathcal{T}$ according to (2). By the assumption $|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}|$, $\forall \mathcal{S} \subseteq \mathcal{X}_\mathcal{T}$, and Lemma 1, we have that g–rank$(C_\mathcal{T}[\bar{A}, \bar{B}]) = |\mathcal{T}|$, which implies that there exists a proper variety $W \subset \mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}}$, such that if $[\mathbf{p}_{\bar{A}}, \mathbf{p}_{\bar{B}}] \in V^c \cap W^c$, then $\operatorname{rank}(C_\mathcal{T}[\tilde{A}, \tilde{B}]) = |\mathcal{T}|$, i.e., $\boldsymbol{N}((C_\mathcal{T}[\tilde{A}, \tilde{B}])^\top) = \mathbf{0}$. Define $\hat{I} \in \mathbb{R}^{n\times n}$ as

$$[\hat{I}]_{ij} = \begin{cases} 1, & \text{if } j = i,\ i \in \mathcal{T} \\ 0, & \text{otherwise.}\end{cases} \quad (19)$$

We claim that there does not exist a nontrivial vector $e \in \mathbb{C}^n$ such that $\hat{I}e = e$, $e^\top\tilde{A} = 0e^\top$ and $e^\top\tilde{B} = \mathbf{0}$. Otherwise, $e^\top[\tilde{A}, \tilde{B}] = \mathbf{0}$, which contradicts $\boldsymbol{N}((C_\mathcal{T}[\tilde{A}, \tilde{B}])^\top) = \mathbf{0}$.

Hence, if $[\mathbf{p}_{\bar{A}}, \mathbf{p}_{\bar{B}}] \in V^c \cap W^c$, then there is no nontrivial vector $v \in \mathbb{C}^{|\mathcal{T}|}$, such that $v^\top C_\mathcal{T} \in \operatorname{span}\{e_1^\top, \ldots, e_l^\top\}$. Thus, generically, $\boldsymbol{N}((C_\mathcal{T} Q(\tilde{A}, \tilde{B}))^\top) = \mathbf{0}$. The $(\bar{A}, \bar{B})$ is structurally target controllable with respect to $\mathcal{T}$. ■

D. Proof of Theorem 2

Proof: ($\Longleftarrow$)Suppose there exists a target set $\mathcal{T} = \{t_i\}_{i=1}^k \subseteq [n]$ such that there is no right-unmatched vertex in $\mathcal{B}(\mathcal{X}_\mathcal{T}, \mathcal{Y}, \mathcal{E}_{\mathcal{X}_\mathcal{T},\mathcal{Y}})$, and $(\bar{A}, \bar{B})$ is structurally target controllable with respect to $\mathcal{T}$. We construct $\tilde{C} \in \{0,1\}^{k\times n}$ such that $[\tilde{C}]_{it_i} = 1$ and $\sum_{j=1}^n [\tilde{C}]_{ij} = 1, \forall i \in [k]$. Since $(\bar{A}, \bar{B})$ is structurally target controllable with respect to $\mathcal{T}$, there exist numerical realizations $\tilde{A}, \tilde{B}$ such that $\tilde{C}\cdot Q(\tilde{A}, \tilde{B})$ is full row rank. Thus, $(\bar{A}, \bar{B}, \bar{C})$ is structurally output controllable. ■

($\Longrightarrow$)We approach the proof by contraposition. Suppose for all target sets $\mathcal{T} \subseteq [n]$ with respect to which $(\bar{A}, \bar{B})$ is structurally target controllable, there exists at least one right-unmatched vertex in $\mathcal{B}(\mathcal{X}_\mathcal{T}, \mathcal{Y}, \mathcal{E}_{\mathcal{X}_\mathcal{T},\mathcal{Y}})$. Then, by taking a similar reasoning used in the proof of Lemma 7, we can show that $\operatorname{rank}(\tilde{C}\cdot Q(\tilde{A}, \tilde{B})) \leq k, \forall \tilde{C} \in \mathbb{R}^{k\times n}$, which implies $(\bar{A}, \bar{B}, \bar{C})$ is not structurally output controllable.

E. Proof of Theorem 3

Proof: The NP-hardness can be proved by reducing a general instance of 3-D matching problem [31, p.46] to an instance of the structural output controllability problem. More specifically, the elements in the three dimensions $\mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{S}_3$ of the three-dimensional matching problem are recast as vertices in $\mathcal{U} \times \mathcal{X} \times \mathcal{Y}$, where $\mathcal{U}, \mathcal{X}$, and $\mathcal{Y}$ are input, state, and output vertices, respectively. The links in $\mathcal{S}_1 \times \mathcal{S}_2$ and in $\mathcal{S}_2 \times \mathcal{S}_3$ are recast as edges in $\mathcal{E}_{\mathcal{U},\mathcal{X}}$, and $\mathcal{E}_{\mathcal{X},\mathcal{Y}}$, respectively. We let $[\bar{A}]_{ij} = 0$, for $\forall i, j \in [|\mathcal{X}|]$; $[\bar{B}]_{ij} = \star$ if $(u_j, x_i) \in \mathcal{E}_{\mathcal{U},\mathcal{X}}$ and $[\bar{B}]_{ij} = 0$ otherwise; $[\bar{C}]_{ij} = \star$ if $(x_j, y_i) \in \mathcal{E}_{\mathcal{X},\mathcal{Y}}$ and $[\bar{C}]_{ij} = 0$ otherwise. By Theorem 2, the constructed structural system $(\bar{A}, \bar{B}, \bar{C})$ is structurally output controllable if there exists a three-dimensional matching in $\mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{S}_3$. Since such a reduction can be completed in polynomial time, the problem of verifying conditions in Theorem 2 is NP-hard.

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